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Abstract

A rigorous treatment is given of two basic types of one-dimensional problems in the theory of the multiple scattering of waves by distributions of point scatterers. The first problem concerns scattering by a single configuration of scatterers with fixed but arbitrary positions in the (finite) scattering interval. The second problem concerns scattering from an ensemble of configurations, the ensemble arising out of the assumption that the scatterer positions are random variables. In each problem the solution u^n (n denotes the number of scatterers) is approximated by the solution U^n of a continuum problem that results when the discrete distribution function $W^n(x)$, characterizing the spatial distribution of scatterer positions, is replaced by a suitable approximating continuous distribution function $W(x)$. In both problems the object is to determine conditions which ensure that u^n is approximated, in an appropriate sense, by U^n.

Our procedure consists basically in deriving estimates of the maximum modulus of i) $u^n - U^n$, ii) u^n minus a sum of terms having U^n as a dominant term and iii) the average of these quantities. These estimates are then employed to obtain sufficient conditions that ensure a small error of approximation. Both the estimates and the sufficient conditions are expressed solely in terms of the basic parameters of the problem - i.e., the number of scatterers, the free space wave number, the length of the scattering region, and the magnitude of the ratio of scatterer impedance to the free space wave impedance. The fact that $W(x)$ is " close " to $W^n(x)$ is also taken into account. Our results provide rigorous justifications, in a one-dimensional setting, for two formal procedures now in wide use.

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1. Introduction

There are two basic problems in the theory of the multiple scattering of radiation by finite collections of elementary scatterers. One problem involves scattering by a single configuration of scatterers with fixed but arbitrary positions in the scattering region; the other, the random-configuration or, as we shall sometimes call it, the statistical problem, involves scattering by an ensemble of configurations. In the latter problem, the ensemble arises out of the assumption that the positions of the scatterers are random variables. For example, in classical optics, where the elementary scatterers are dipoles, the scattering of light waves by an isotropic substance like glass is an instance of a single configuration problem. On the other hand, the scattering of light waves by a gas of dipoles is an example of a random configuration problem. Statistics enter into this problem when time averages are identified with configurational averages.*

In this report, we present a rigorous treatment of one-dimensional single- and random-configuration problems. The scattering elements are assumed to be identical non-dissipative point scatterers. It is further assumed that they are completely characterized by an impedance function and that they are microscopic in the sense that the ratio of the scatterer impedance to the free space wave impedance is much less, in absolute value, than unity. In each of the two problems the solution u^n (n denotes the number of scatterers) is approximated by the solution of a continuum problem U^n that results when the discrete distribution function, characterizing the spatial distribution of scatterer positions, is replaced by a suitable approximating continuous distribution function. In both problems the object is to determine conditions which ensure that u^n is approximated, in an appropriate sense, by U^n. The relation between the two problems is also discussed.

The statistical problem is treated essentially by reducing it to the single-configuration problem. In both problems physically reasonable sufficient conditions are given which ensure a close approximation of the discrete to the continuum problem and estimates of the error of approximation are developed. These sufficient conditions and estimates, it should be added, depend only on the basic parameters of the scattering process, namely, the number of

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* We shall be concerned exclusively with configurational averages in the present work.

scatterers, the free space wave number, the length of the scattering region, the magnitude of the ratio of the scatterer impedance to the free space wave-impedance and finally on a suitable measure of how closely the continuum distribution function approximates the discrete distribution function. Our treatment is completely general in the following sense: (1) Neither the precise form of the incident radiation nor the precise form of the continuum distribution function plays an essential role in obtaining the estimates mentioned above. (2) The solution of the continuum problem need not be known explicitly. (3) In the statistical problem, the scatterer positions are required only to be identically distributed random variables - they need not be statistically independent. (4) Finally, we treat not only the case where n is fixed, but also the classical limiting case where the scatterer impedance is proportional to $1/n$ and n is allowed to become infinite.

It may be mentioned that our results apply to certain natural one-dimensional problems that occur in practice. These problems have to do with the construction of artificial dielectrics in wave guides and transmission lines (electrical and acoustical). In the language of transmission line theory, the scattering elements are small lumped impedances or admittances placed along the transmission line. Our results provide a basis for deciding to what degree collections of such elements may be regarded as forming an artificial (continuous) dielectric. As they stand, the results apply either to scattering by impedances arranged in series or to admittances arranged in parallel. It is an easy matter, however, to extend them to include parallel-series arrangements.

There exists an extensive literature of formal treatments of the foregoing problems - often, in a three-dimensional setting. Many of these have been reviewed by Lax [1]-a),-b),; we may therefore refer the reader to Lax's papers for bibliographical material.* Here, we shall confine ourselves to discussing two works which bear directly on the present investigation. The first is Born's [4] treatment of the scattering of light waves by dipoles having fixed positions in space. In this work Born employs the widely-used procedure of

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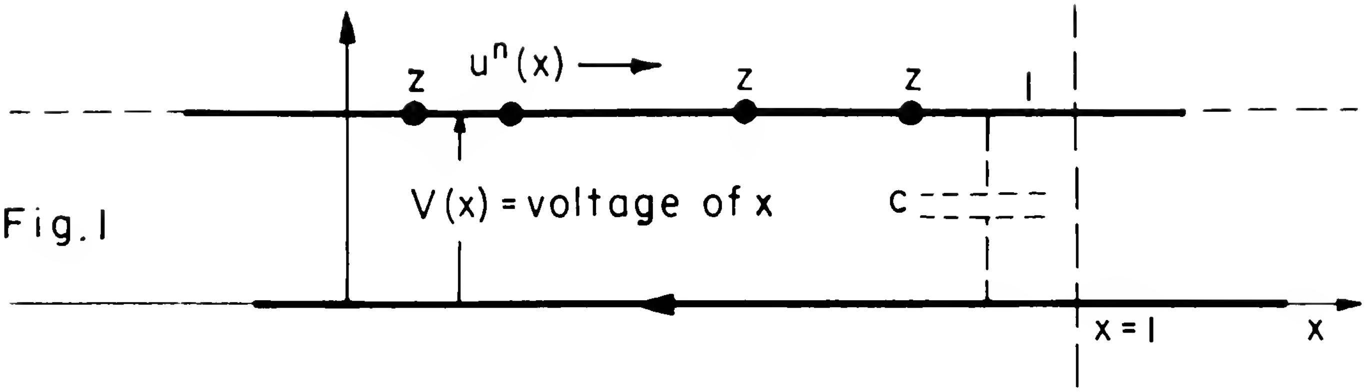
*An early work, not mentioned by Lax, dealing with the scattering of electromagnetic waves by a gas of dipoles, is due to Reiche [2]. Additional bibliographical material will be found in a report [3] of Twersky where Foldy's procedure is applied to the scattering of scalar waves by volume distributions of parallel cylinders.

replacing the discrete distribution of scatterers at the very outset by a suitably smoothed-out distribution. Physical reasoning suggests that this procedure is a valid one provided the scattering region can be divided into subregions whose linear dimensions are small in comparison with the wavelength and yet contain a large number of scatterers. Our treatment of the single-configuration problem may be regarded as providing a justification of this reasoning in one-dimensional problems.

A systematic formal procedure for dealing with random configuration problems is due to Foldy [5] who discussed the scattering of scalar waves by isotropic monopole scatterers. Foldy's work has been extended by Lax, in the works cited above, to include vector waves scattered by dipoles as well as by monopoles and in several other directions. We shall sketch Foldy's procedure as applied to one-dimensional problems later. Here, we need only say that our results for the statistical problem provide a justification for the final results of the Foldy procedure.

It should be stressed that the formal procedures just mentioned apply equally well to one- and higher-dimensional problems. In limiting our discussion to one-dimensional problems several important analytical difficulties are avoided. The ultimate source of these difficulties is the fact that the fields of the individual scatterers become infinite at the position of the scatterer. It is essentially this fact which necessitates the use of the so-called 'self-consistent' approach in two and three-dimensional problems. The one-dimensional problems, however, still retain many of the essential features of their higher dimensional analogs. Moreover, it appears that the present analysis may be extended to include suitably reformulated versions of the higher dimensional problems. This extension will be the subject of a future report.



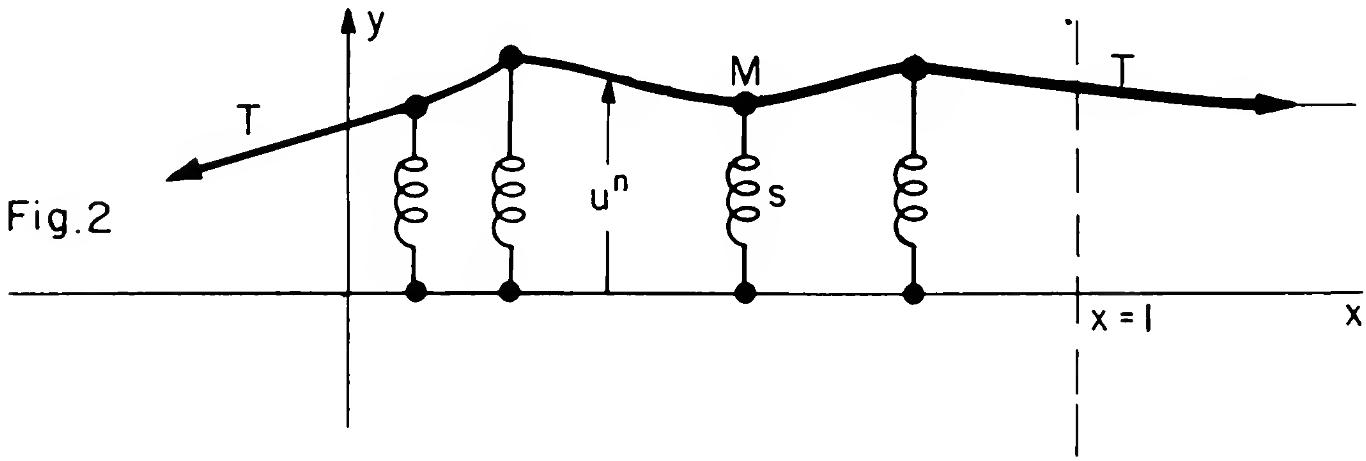


Fig. 1 The transmission line model. The function $u^n(x)$ represents the current in the lines. The quantities ℓ and c are the distributed series inductance and shunt capacitance, respectively, per unit length of line. In the model the z's are small lumped impedances and $Z = \sqrt{\ell/c}$.

Fig. 2 The mechanical model. The function $u^n(x)$ is displacement of an infinite string which has a uniform mass density ρ and is under tension T. The identical scatterers consist of discrete masses M embedded in the string that are attached to springs of spring constant s that are in turn attached to the x-axis. Here, $Z = (T\rho)^{1/2}$ and $z(k) = i(\rho/T)^{1/2}\left[(s/k)-kTM/\rho\right]$.

PART I. THE SINGLE CONFIGURATION PROBLEM

2. Formulation of the problem

The basic equation of the scattering process will be expressed in the following form:*

$$(2.1)\qquad u^n_{xx} + \left[k^2 + (ikz/Z)\sum_{q=1}^{n}\delta(x-x_q)\right]u^n = 0, \qquad -\infty < x < \infty .$$

Here, n is the number of scatterers, $\delta(x)$ the Dirac delta function. The quantity x_q denotes the position of the q-th scatterer; it will be assumed that

$$(2.2)\qquad 0 \leq x_q \leq L, \qquad\qquad q = 1,2,\ldots,n,$$

L being the length of the scattering region. The quantity $k = 2\pi/\lambda$, where λ is the wavelength, is the free-space wave number. Z is the characteristic impedance of the medium when no scatterers are present; Z is normally a positive quantity and independent of the frequency or, alternatively, the wave number of the incident radiation. The quantity $z = z(k)$ represents the impedance of the (identical) individual scatterers. It will be assumed that these scatterers are non-dissipative and 'microscopic'. These requirements may be simply expressed as follows:

$$(2.3)\qquad z \text{ is purely imaginary,}$$

$$(2.4)\qquad \frac{|z|}{Z} \ll 1.$$

The quantity u^n is the total wave function resulting from the scattering of incident radiation—$u_o(x)$ say—by the n scatterers. The wave function $u^n(x)$ is required to i) be continuous for all x, $-\infty < x < \infty$, ii) be expressible as a superposition,

$$(2.5)\qquad u^n = u_o(x) + v^n(x),$$

*Equation (2.1) will be recognized as an abbreviated way of writing the two equations $u^n_{xx} + k^2u^n = 0$, $x \neq x_q$ and $u^n_x\big|_{x=x_q+0} - u^n_x\big|_{x=x_q-0} = -\frac{iknz}{Z}u^n(x_q)$, $q = 1,2,\ldots,n$.

of the incident radiation $u_o(x)$ and a scattered field $v^n(x)$, and finally iii) satisfy (2.1).

For the sake of simplicity, the incident radiation will be specialized to the plane wave of unit amplitude,

$$(2.6) \qquad u_o = \exp(ikx),$$

incident on the scattering region from the left.* More general types of incident radiation may be treated simply by invoking the principle of superposition and making obvious changes in the ensuing analysis.

In the language of transmission line theory $u^n(x)$ is the current at x in a non-dissipative line (see Figure 1). The scatterers, as mentioned earlier, may be thought of as small lumped impedances arranged in series along the line at $x = x_q$, $q = 1,2,\ldots,n$. A fuller discussion of this scattering model will be found in Appendix I.A.

Another simple 'concrete' model of the scattering process is an infinite homogeneous string which is subject to a constant tensile force and is allowed to vibrate in a direction transverse to the x-axis (see Figure 2). In this model the identical scatterers are point masses attached to small springs of a given stiffness; the point masses are embedded in the string and the remaining end of each spring is attached to the x-axis. A detailed analysis of this system is given in Appendix I.B.

In Section 1 we spoke of replacing the " discrete distribution function**" by " continuous d.f." and assigning to $u^n(x)$ by this means the solution U^n of a " continuum problem". We turn now to the task of formulating precisely what we shall mean by these terms.

Let us write

$$(2.7) \qquad H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} .$$

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* The time dependence $\exp(-i\omega t)$ will be assumed throughout.

** The expression "distribution function" will hereafter be abbreviated as d.f..

Then $W^n(x)$ ($= W^n(x;\ x_1,\ldots,x_n)$), the ratio $\nu_n(x)/n$ of the number $\nu_n(x)$ of scatterers in the closed interval $[0,x]$ to the total number n of scatterers may be expressed as follows:

$$(2.8)\qquad W^n(x) = \left(\sum_{q=1}^{n} H(x-x_q)\right)\Big/ n.$$

Hereafter, $W^n(x)$ will be referred to as the <u>discrete d.f.</u> of the n scatterer positions. By a <u>continuum</u> d.f. $W(x)$ we shall mean a d.f. which may be represented as follows:

$$(2.9)\qquad W(x) = \int_0^x w(\xi)d\xi .$$

It is assumed here that $w(x)$ is a continuous probability density function in the scattering region, $0 \leq x \leq L$; specifically that $w(x)$ i) is a non-negative everywhere continuous function of x except perhaps at $x = 0$ or L where finite jumps are allowed, ii) vanishes outside the scattering region and iii) has the property

$$(2.10)\qquad \int_0^L w(\xi)d\xi = 1.$$

Let ϵ be any positive number. Let us say that a continuum d.f. has or satisfies the property $\mathscr{P}(\epsilon)$ with respect to the discrete d.f. $W^n(x)$ if

$$|W^n(x) - W(x)| \leq \epsilon , \qquad -\infty < x < \infty .$$

Evidently, $W(x)$ is not uniquely determined by this property. In fact, if $\epsilon = 1$ <u>all</u> continuum d.f.'s have the property $\mathscr{P}(\epsilon)$ with respect to $W^n(x)$. We shall be interested chiefly in the case where ϵ is small; nevertheless, any $W(x)$ having the property $\mathscr{P}(\epsilon)$, $\epsilon \leq 1$, could be regarded as a suitable 'continuous d.f.' and our analysis completed without additional assumptions. We wish, however, to consider not only the case where n is fixed and $W(x)$ has the property $\mathscr{P}(\epsilon)$, but also the classical limiting case in which i) $|z/Z| = \rho_o/n$, ρ_o being a constant independent of n and ii) n is allowed to become infinite in such a way that $\lim_{n\to\infty} W^n(x) = W(x)$. Moreover, we wish to be able to treat both cases within the same analytical framework. This goal may be achieved by reformulating $\mathscr{P}(\epsilon)$ as follows:

To begin, we consider not a single problem with fixed n but an infinite sequence of related problems such that in the j-th problem there are $n(j) \geq j$, $j = 1,2,\ldots,$ scattering elements. Let $\mathcal{C}^j = \{x_1, x_2, \ldots, x_{n(j)}\}$ denote the j-th configuration of scatterer positions in the corresponding sequence $\mathcal{C}^1$, $\mathcal{C}^2, \ldots,$ of scatterer configurations. It will be assumed that $\mathcal{C}^j$ in this sequence is obtained from $\mathcal{C}^{j-1}$ simply by adding $n(j) - n(j-1)$ scatterer positions; the labelling of the $n(j-1)$ elements of $\mathcal{C}^{j-1}$ is to remain unchanged. Under these circumstances, the positions x_q, $q = 1,2,\ldots,n(j)$, of the scatterers in $\mathcal{C}^j$ may be thought of as the first $n(j)$ elements of an infinite sequence x_q, $q = 1,2,\ldots,$ formed from the elements of $\mathcal{C}^j$, $j = 1,2,\ldots,$ in the natural way - that is, with the original labelling.

Consider the infinite sequence of d.f.s

$$(2.11) \qquad W^{n(j)}(x) = \frac{1}{n(j)} \sum_{q=1}^{n(j)} H(x-x_q), \qquad j = 1,2,\ldots \quad .$$

This sequence will be said to <u>satisfy the property $\mathcal{P}$</u> (the analog of $\mathcal{P}(\epsilon)$ above) <u>or to have the property $\mathcal{P}$</u> if there exists a continuum d.f. $W(x)$ and a null sequence* $\epsilon(j)$, $j = 1,2,\ldots,$ whose terms are independent of x such that for all x

$$(2.12) \qquad | W^{n(j)}(x) - W(x)| \leq \epsilon(j), \qquad j = 1,2,\ldots \quad .$$

We shall also say that <u>$W(x)$ has or satisfies the property $\mathcal{P}$ with respect to the sequence $W^{n(j)}(x)$, $j = 1,2,\ldots$</u> .

In terms of the sequence x_q, $q = 1,2,\ldots,$ property $\mathcal{P}$ simply requires that the scatterer positions become everywhere dense in the unit interval in such a manner that the relative number of scatterers in $[0,x]$ approaches the limit d.f. $W(x)$ uniformly. There is no essential loss of generality in requiring the convergence to be uniform since this follows automatically. Specifically**, if a sequence of d.f.s is pointwise convergent to a continuous d.f. the convergence is uniform. This fact seems first to have been proved by Polya [6].

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* For all j, it may be assumed that $\epsilon(j) \leq 1$ since $|W^{n(j)} - W(x)|$ is always at most unity.

** I would like to express my gratitude to H.N. Shapiro for bringing this useful fact to my attention.

The following are two concrete examples of sequences having the property described above:

Example (1)*

Let us place the first scatterer at $x_1 = 1$; call this the zeroth refinement of the unit interval. Second, set $x_2 = 1/2$ and call the pair (x_1, x_2) the first refinement at the unit interval. Third, add the midpoints of the intervals $(0,1/2)$ and $(1/2,1)$; these points and x_1 and x_2 form the second refinement. Let $n(j)$ be the number of points at the j-th refinement. Then it is easily verified that

$$(2.13) \qquad n(j) = 2^j$$

and

$$(2.14) \qquad |W^{n(j)}(x) - x| \leq 2^{-j} = \frac{1}{n(j)}, \qquad 0 \leq x \leq 1.$$

Thus at the j-th refinement $W^{n(j)}(x)$ approximates the 'uniform' d.f. $W(x) = x$, with the ϵ - accuracy $n(j)^{-1}$ and converges uniformly to x, $0 \leq x \leq 1$, as the number of refinements become infinite.

Example (2)

In this example $n(j) = j$. Let γ be an irrational number and let the position of the q-th scatterer be given by

$$(2.15) \qquad x_q = \left[(q-1)\gamma\right] \bmod .1,$$

where x_q is the smallest number in the unit interval congruent to $(q-1)\gamma$. For such sequences it is known that $W^n(x)$ approaches the uniform distribution $W(x) = x$ as a limit. If, in addition, γ is a quadratic irrational - that is, if γ is an irrational number satisfying a quadratic equation with rational coefficients - it can be shown that [7]**

$$(2.16) \qquad |W^n(x) - x| \leq O\left[\log n/n\right].$$

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* In our examples we shall always assume that $L = 1$.

** The constant implicit in this estimate may be obtained from the periodic continued fraction representation of γ.

We turn now to the task of defining what we mean by the "continuum problem". To this end we note that the solution u^n of (2.1) satisfies the integral equation

$$u^n(x) = \exp(ikx) - (z/2Z)\int_0^L \exp(ik|x-x'|)u^n(x')\left(\sum_{q=1}^{n} \delta(x'-x_q)\right) dx'.$$

If the term $\sum_{q=1}^{n} \delta(x'-x_q)$ is replaced by the equivalent expression $ndW^n(x')$, where $W^n(x')$ is defined as in (2.8) or (2.11), and the integral is interpreted as a Riemann-Stieltjes integral, then this equation may be rewritten as follows:

$$(2.17) \qquad u^n(x) = \exp(ikx)-(nz/2Z)\int_0^L \exp(ik|x-x'|)u^n(x')dW^n(x').$$

Now let us suppose that $W(x)$ satisfies $\mathcal{P}$. Let us replace $W^n(x)$ in (2.17) by the smoothed-out continuum d.f. $W(x)$ [see (2.9)] and call the new u^n, $U^n(x)$. Evidently $U^n(x)$ satisfies the integral equation

$$(2.18) \qquad U^n(x) = \exp(ikx)-(nz/2Z)\int_0^L \exp(ik|x-x'|)U^n(x')dW(x)$$

or, since $dW(x') = w(x')dx$,

$$(2.19) \qquad U^n(x) = \exp(ikx)-(nz/2Z)\int_0^L \exp(ik|x-x'|)U^n(x')w(x')dx.$$

The solution* of (2.19) shall be termed the solution of the <u>associated continuum problem</u>. More explicitly, let $W^n(x) = W^{n(j)}(x)$, $j = 1,2,\dots$, be a sequence of d.f.s corresponding to the sequence $\mathcal{C}^j$ introduced earlier. Let $W(x)$ be the continuum d.f. which enjoys the property $\mathcal{P}$ with respect to this sequence. Then by definition, <u>$U^n = U^{n(j)}$</u>, the solution of (2.19) <u>is for each j the solution of the continuum problem that we associate with $u^{n(j)}$</u>.

It is a simple matter to verify, using (2.19), that i) U^n is a superposition of the incident plane wave $\exp(ikx)$ and a radiating wave function and ii) U^n is an everywhere continuous function satisfying the differential equation

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*Here and hereafter by 'solution' we shall mean <u>continuous</u> solution.

$$(2.20) \qquad U^n_{xx} + \left[k^2 + (iknzw(x)/Z)\right] U^n = 0, \qquad -\infty < x < \infty .$$

From the general theory of scattering in one dimension it is known (see [8]) that U^n exists and is unique. In this connection, it should be mentioned that the general theory also implies the unique existence of u^n, the solution of (2.1)*.

Heuristic considerations of the type outlined earlier (see Section 1) in connection with Born's[4] treatment of the scattering of electromagnetic waves suggest that U^n is in some sense an approximation to u^n. We shall now take the first step toward attaching a precise meaning to this statement. This will also enable us to give an accurate statement of our basic objectives and to complete the formulation of the problem. For this purpose, it is first necessary to introduce the radiating Green's function $G^n(x,x')$ of the continuum problem: $G^n(x,x')$ is a continuous function of x,x', $-\infty < x,x' < \infty$ which i) has continuous x-derivatives everywhere except at $x = x'$, ii) obeys the symbolic equation

$$(2.21) \qquad \frac{d^2G^n}{dx^2} + \left[k^2 + \frac{iknz}{Z} w(x)\right] G^n = -\delta(x-x'),$$

and iii) satisfies the outgoing wave condition. The explicit form (see subsection 6.21) need not concern us here; for our purposes we need only say that, as with U^n, the general theory of one-dimensional scattering implies the unique existence of G^n.

Now let $f(x)$ be any function continuous in the interval $0 \leq x \leq L$. Define the operator T_n as follows:**

$$(2.22) \qquad T_n f(x) \equiv \int_0^L 2ik\, G^n(x,x')f(x')d\left[W^n(x') - W(x)\right] .$$

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* Actually, the 'existence' part of the theory is not essential here since an explicit expression for $u^n(x)$ may be exhibited (see subsection (6.22)). It must be added, however, that this expression is too complicated to be of use in the present considerations.

**The upper and lower limits of integration are to be interpreted respectively as 0-0 and L+0 - thus jumps, if any, of $W^n(x)-W(x)$ that occur at $x = 0$ or $x = L$ are included.

Then it is easy to show that u^n satisfies the integral equation

(2.23) $$u^n = U^n(x) + \frac{nz}{2Z} T_n u^n(x).$$

The simplest way of proving this statement is to operate on both sides with $d^2(\ )/dx^2 + [k^2+(iknzw(x)/Z)]$ and use (2.22), (2.21) and (2.20) to prove that $u^n(x)$ satisfies (2.1). It is also necessary to check that the expression for u^n in (2.23) is a sum of the incident plane wave exp(ikx) and an outgoing wave; but this follows immediately from the definition of U^n and the fact that $T_n u^n(x)$ is an outgoing wave.

On iterating (2.23) j times we find that

(2.24) $$u^n(x) = \sum_{\nu=0}^{j} \left(\frac{nz}{2Z}\right)^\nu T_n^\nu U^n(x) + \left(\frac{nz}{2Z}\right)^{j+1} T_n^{j+1} u^n(x).$$

Here, we have set

(2.25) $$T_n^0 f(x) = f(x)$$

and

(2.26) $$T_n^\nu f(x) = T_n(T_n^{\nu-1} f(x)), \qquad \nu = 1,2,\ldots .$$

Finally, we find that

(2.27) $$u^n = \sum_{\nu=0}^{\infty} \left(\frac{nz}{2Z}\right)^\nu T_n^\nu U^n(x)$$

provided the infinite sum converges.

We are now in a position to state our basic objectives. Let us define the norm, $||f||$, of any function f(x), continuous in the infinite interval $-\infty < x < \infty$, as follows:

(2.28) $$||f|| = \underset{-\infty<x<\infty}{\text{Maximum}} |f(x)| .$$

Then we wish to find sufficient conditions which will enable us to decide i) when the relative errors R_n^p, $\tilde{R}_n^p$ and the absolute error A_n^p of order p; namely,

$$R_n^p \equiv \left(\| u^n - \sum_{\nu=0}^{p} \left(\frac{nz}{2Z}\right)^{\nu} T_n^{\nu} U^n \|\right) / \| u^n \| = \left(\frac{n|z|}{2Z}\right)^{p+1} \frac{\| T_n^{p+1} u^n \|}{\| u^n \|} ,$$

$$\text{(2.29)} \qquad \tilde{R}_n^p \equiv \left(\| u^n - \sum_{\nu=0}^{p} \left(\frac{nz}{2Z}\right) T_n^{\nu} U^n \|\right) / \| U^n \| = \left(\frac{n|z|}{2Z}\right)^{p+1} \frac{\| T_n^{p+1} u^n \|}{\| U^n \|} ,$$

$$A_n^p = \left(\| u^n - \sum_{\nu=0}^{p} \left(\frac{nz}{2Z}\right) T_n^{\nu} U^n \|\right) = \left(\frac{n|z|}{2Z}\right)^{p+1} \| T_n^{p+1} u^n \| ,$$

are small and finally, ii) when the series expression for $u^n(x)$ in (2.27) converges.

The quantities R_n^p, $\tilde{R}_n^p$ and A_n^p are measures of the degree to which $u^n(x)$ is approximated by the finite sum

$$\sum_{\nu=0}^{p} (nz/2Z)^{\nu} \; T_n^{\nu} \; U^n(x).$$

In the present problem, the case $p = 0$ is perhaps the only one of interest; here the finite sum reduces to the single term $U^n(x)$. We shall, however, carry out all calculations for arbitrary values of p. There are two reasons for doing so. First, and most important, these calculations will be used in our treatment of the random configuration problem. Second, and quite apart from the considerations of this report, sums of this type (or convergent sums - see (2.23)) can, when n is large, provide a good approximation to u^n while, under the same conditions, the explicit solution would be totally unmanageable.

We shall require of our sufficient conditions that they be expressible solely in terms of the basic physical parameters of the problem; namely, n, z, Z and k and L. Within this framework, let us examine briefly some of the possible ways of making the relative error R_n^p, for example, small. First, R_n^p may be made small by making $(n|z|/2Z)$ small in comparison with unity. Second the possibility exists of making the 'norm' T_n^{p+1} $(= \| T_n^{p+1} u_n \| / \| u_n \|)$ small even when $n|z|/2Z$ exceeds unity. This possibility arises out of the fact that $W^n(x)$ is "close" to $W(x)$ when n is large [see (2.12)]. These rough arguments suggest that we should look for sufficient conditions which take into account both ways of making R_n^p small. Such conditions may indeed be obtained; they will be formulated and discussed in the next section.

3. Survey of results

3.1 Introductory remarks. In this section we shall present the main results of Part I. These results consist chiefly of estimates of the errors, R_n^p, $\tilde{R}_n^p$ and A_n^p defined in (2.29), and of sufficient conditions which are based on them. We shall merely summarize the estimates here - proofs will be given in Part III. Our presentation of the sufficient conditions will, however, be entirely self-contained.

Our results will be expressed in terms of the following quantities:

$$r = |z|/Z, \tag{3.1}$$

$$\alpha_n = nr\left[1 + (nr/2)\exp(nr)\right] \text{Max.}\left\{1, (nr/2)\right\}, \tag{3.2}$$

$$\beta_n = \text{Min.}\left\{1,\ 6\sqrt{2\epsilon(n)\text{Max}\left\{1, kL\right\}}\right\}. \tag{3.3}$$

Here, by Max. $\{1,a\}$ we mean 1 or a, whichever is larger, and by Min. $\{1,a\}$ 1 or a, whichever is smaller.

Until now, we have allowed the number of scatteres in the j-th configuration $\mathcal{C}^j$ to exceed or equal j. In order to simplify the statement of our results we shall hereafter assume j = n - i.e., that $\mathcal{C}^n$ contains n scatterers. No difficulty will be encountered in extending the stated results to cover the more general situation.

To avoid repetition, let it once and for all be assumed that (i) a sequence of discrete d.f.'s $W^n(x)$, n = 1,2,..., having the property $\mathcal{P}$ has been specified;(ii) W(x) is the continuum d.f. associated with this sequence; (iii) a definite null sequence $\epsilon(n)$, n = 1,2,..., has been given such that $\epsilon(n) \geq |W^n(x) - W(x)|$, $0 \leq x \leq L$; and finally, (iv) the function $u^n(x)$ associated with $W^n(x)$ and $U^n(x)$ with W(x) are the (unique) solutions of the integral equations (2.17) and (2.19), respectively. Our basic estimates are summarized in the following:

Theorem (1). Let the relative and absolute errors of order p, R_n^p, $\tilde{R}_n^p$ and A_n^p be defined as in (2.29); then for any p, p = 0,1,2,..., R_n^p, $\tilde{R}_n^p$ and A_n^p satisfy the following inequalities:

$$R_n^p \leq \left[1 + (nr/2)\right] (\alpha_n\beta_n)^{p+1}, \tag{3.4}$$

$$(3.5)\qquad \tilde{R}_n^p \leq \left[1 + (nr/2)\right](\alpha_n\beta_n)^{p+1},$$

$$(3.6)\qquad A_n^p \leq \left[1 + (nr/2)\right](\alpha_n\beta_n)^{p+1}\exp(nr),$$

where r, α_n and β_n are defined as in (3.1),(3.2) and (3.3), respectively.

Theorem (1) is arrived at by finding estimates for the quantities $\|T_n^{p+1}u^n\|/\|u^n\|$, $\|T_n^{p+1}U^n\|/\|U^n\|$, (subsection 6.3 - Part III), $\|u^n\|$ and $\|U^n\|$ (subsection 6.2 - Part III) and combining these estimates in the obvious way. Specifically, Theorem (1) follows immediately from (6.50), (6.49), the definitions (6.47), (6.39), (6.38) and the estimates (6.2), (6.1) and (6.24). It is not claimed that (3.4)-(3.6) furnish the best possible estimates under all circumstances. In fact, in the special case in which $W(x) = x$, $0 \leq x \leq L$, and* $iz > 0$, it can be shown that the exponentials in (3.2) and (3.6) may be replaced by unity. We shall not, however, make use of this fact in the sequel since we are interested solely in general results.

3.2 <u>The sufficient conditions</u>. It is convenient to distinguish three cases. We shall assume in Case (1) that $r = \rho_o'/n$, where ρ_o' is any positive number,** and then ascertain the consequences of letting n become infinite. Earlier we referred to this case as the classical limiting case; our reason for doing so will soon be made clear. In the remaining two cases, it will be assumed that r is independent of n; the product nr will, however, be required to satisfy the relation $0 \leq nr \leq \rho_o$, where ρ_o is any, as yet unspecified, positive number. Cases (2) and (3) will be distinguished by requiring that n is "large" or "moderate" (Case (2)) and that n is "small" (Case (3)). Here, if n_o is the largest integer satisfying the inequality

$$(3.7)\qquad 6\sqrt{2kL\epsilon(n)} > 1,$$

then n is called "large" or "moderate" if n exceeds n_o; otherwise n is said to be "small". Observe that n is large or moderate if, and only if, $\beta_n < 1$ and small if, and only if, $\beta_n \geq 1$ [see (3.3)]. To be sure, the present separation into two cases according as n is small or not is somewhat arbitrary; it is nevertheless a natural one for the purposes of our discussion.

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* It will be recalled that z is a pure imaginary.

** ρ_o' is assumed to be independent of n.

We shall now discuss each of the three cases in greater detail.

Case (1): $r = \rho_o'/n$

We are dealing here with the classical case of the passage from a discrete distribution of scatterers to a continuum distribution. Consider, for example, the mechanical model of the scattering process mentioned in Section 1 (also see Figure 2 and Appendix I.B). For the sake of simplicity let us suppose that the spring constant of the string vanishes and that M, the mass of each scatterer, is inversely proportional to n - specifically that $M = \sigma/n$. It is reasonable to expect that, as n approaches infinity, the total mass density of the string will approach $\rho + \sigma$ and that $u^n(x)$ will approach the solution of the continuum problem having $\rho + \sigma$ for the total mass density in the scattering region*. This result can be rigorously justified - it is, in fact, a special case of a theorem which we shall now state and prove.

Theorem (2). Let it be assumed that $r = \rho_o'/n$, where $\rho_o' > 0$ is independent of n. Then, i) the continuum solution U^n is independent of n (we shall henceforth write U instead of U^n), ii) U is the (unique) solution of the integral equation**,

$$U(x) = \exp(ikx) + \frac{i\rho_o'}{2}\,\mathrm{sgn}(iz)\int_0^L \exp(ik|x-x'|)U(x')w(x')dx' \tag{3.8}$$

and iii)

$$\lim_{n\to\infty} u^n(x) = U(x), \tag{3.9}$$

uniformly for all x in the infinite interval $-\infty < x < \infty$.

We begin the proof by noting that i) and ii) follow immediately from (2.19) on substituting $-i\rho_o'\,\mathrm{sgn}(iz)$ for nz/Z. It is therefore sufficient to prove that

$$\lim_{n\to\infty} ||u^n - U|| = 0. \tag{3.10}$$

To this end, we observe that (3.6) becomes, after setting $p = 0$ and $nr = \rho_o'$,

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* We are assuming here that the final as well as the original mass density in the scattering region is uniform.

**
$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

(3.11) $$\| u^n - U \| \leq \rho_o' \left[1 + (\rho_o'/2)\right] \exp(\rho_o')\alpha_n(\rho_o')\beta_n .$$

Equation (3.10) now follows immediately from this relation, equation (3.3) and the fact that $\epsilon(n)$, $n = 1,2,\ldots$, is a null sequence.

The function U in (3.8) is clearly a superposition of an incoming plane wave and a radiating part and satisfies the differential equation (cf.(2.20))

$$\frac{d^2U}{dx^2} + \left[k^2 + k\rho_o' \operatorname{sgn}(iz)w(x)\right] U = 0.$$

Returning to the case of the vibrating string we can easily show (see Appendix I.B) that $k^2 = \omega^2\rho/T$, $\rho_o' = k\sigma/\rho$ and $\operatorname{sgn}(iz) > 0$ where T is the tension of the string. If $w(x)$ is assumed to be equal to $1/L$ within the scattering region then the above equation reduces to

$$T\frac{d^2U}{dx^2} + \omega^2(\rho + \sigma)U = 0 \qquad 0 \leq x \leq L,$$

and

$$T\frac{d^2U}{dx^2} + \omega^2 \rho U = 0, \qquad x < 0, \quad \text{or} \quad x > L.$$

This is the result we anticipated earlier.

<u>Case (2)</u>: $nr \leq \rho_o$, n moderate or large

Hereafter, we shall employ I_{ρ_o} to denote the interval $0 \leq nr \leq \rho_o$ - i.e.,

$$I_{\rho_o} = \left\{ nr \mid 0 \leq nr \leq \rho_o \right\}.$$

The main results for Case (2) are contained in the following Theorems:

<u>Theorem (3)</u>: Let nr be any point of I_{ρ_o}. Then in order that the right-hand side of

(3.12) $$u^n(x) = \sum_{\nu=0}^{\infty} (nz/2Z)^{\nu}\, T_n^{\nu}\, U^n(x)$$

may converge to the left-hand side uniformly in x in the infinite interval $-\infty < x < \infty$, it is sufficient to require that kL, n, nr and ρ_o satisfy the following relation:

$$(3.13) \qquad (nr)^2\epsilon(n)\ \mathrm{Max.}\left\{kL,\ 1\right\} \le \frac{1}{2}\left(\frac{\rho_o}{6\alpha_n(\rho_o)}\right)^2, \quad nr \in I_{\rho_o}.$$

Theorem (4). Let p, the order of approximation, be fixed and let $\delta^{(s)}$, $0 < \delta^{(s)} < 1$, $s = 1,2,3$, be any three preassigned positive numbers. Let $\theta_p^{(s)}$, $s = 1,2,3$ be defined as follows:

$$(3.14) \qquad \begin{aligned} \theta_p^{(s)} &= \left[\frac{\delta^{(s)}}{1+(\rho_o/2)}\right]^{\frac{1}{p+1}}, \qquad s = 1,2, \\ \theta_p^{(3)} &= \left[\frac{\delta^{(3)}}{1+(\rho_o/2)}\right]^{\frac{1}{p+1}} \exp(-\rho_o/p+1). \end{aligned}$$

Then, for all $nr \in I_{\rho_o}$, the relative and absolute errors R_n^p, $\tilde{R}_n^p$ and A_n^p will satisfy the relations

$$(3.15) \qquad \begin{aligned} R_n^p &\le \delta^{(1)}, \\ \tilde{R}_n^p &\le \delta^{(2)}, \\ A_n^p &\le \delta^{(3)}, \end{aligned}$$

if the quantities n, kL, ρ_o and $nr(< \rho_o)$ satisfy

$$(3.16) \qquad (nr)\epsilon(n)\ \mathrm{Max.}\left\{kL,\ 1\right\} \le \frac{1}{2}\left[\frac{\rho_o\theta_p^{(s)}}{6\alpha(\rho_o)}\right]^2$$

for s = 1,2, and 3, respectively.

The proofs of Theorems (3) and (4) follow almost immediately from the estimates of Theorem (1). For example, (3.13) implies the relation $\alpha_n\beta_n < 1$ which, in virtue of (3.6), is a sufficient condition for the convergence (3.12).

Because $\epsilon(n)$, $n = 1,2,\ldots$, is a null sequence it is always possible to satisfy (3.13) or (3.16) $s = 1,2,3$ by choosing n sufficiently large - n' say. But, since ρ_o is fixed, this choice implies a restriction on the magnitude of r, namely, $r < \rho_o/n'$. If r is specified beforehand then (3.13) or (3.16), $s = 1,2,3$, can in general be satisfied only if n' is moderate or large and at the same time not too large - that is, n' must satisfy $n' < \rho_o/r$.

When $\rho_o \ll 1$ we find, by employing the definition of $\alpha_n(\rho_o)$ [see (3.2), (3.13) and (3.16)], that

$$(3.17) \qquad (nr)^2\,\epsilon(n)\text{Max.}\left\{kL,1\right\} \leq \frac{1}{72} + O(\rho_o^2), \qquad nr \in I_{\rho_o},$$

$$(3.18) \qquad (nr)^2\,\epsilon(n)\text{Max.}\left\{kL,1\right\} \leq \frac{\left[\delta^{(s)}\right]^{\frac{2}{p+1}}}{72} + O(\rho_o^2), \qquad nr \in I_{\rho_o}.$$

By choosing ρ_o sufficiently small, it is clear that we can satisfy these relations without placing any restriction on the factor $\epsilon(n)\text{Max}\{kL,1\}$ apart from the one which is basic to the present case, namely, that n is moderate or large.

Let us now consider the case of greatest interest: r 'moderate'. For the sake of definiteness let us take p = 0 and $\rho_o = 2$. Then (3.13) and (3.16), s = 1,2,3 will certainly be satisfied, respectively, if

$$(3.19) \qquad (nr)^2\,\epsilon(n)\text{Max.}\left\{kL,1\right\} \leq \frac{1}{2}\left(\frac{1}{55}\right)^2,$$

$$(3.20) \qquad (nr)^2\,\epsilon(n)\text{Max.}\left\{kL,1\right\} \leq \frac{1}{2}\left(\frac{\delta^{(1)}}{110}\right)^2, \qquad s = 1,$$

$$(3.21) \qquad (nr)^2\,\epsilon(n)\text{Max.}\left\{kL,1\right\} \leq \frac{1}{2}\left(\frac{\delta^{(2)}}{110}\right)^2, \qquad s = 2,$$

$$(3.22) \qquad (nr)^2\,\epsilon(n)\text{Max.}\left\{kL,1\right\} \leq \frac{1}{2}\left(\frac{\delta^{(3)}}{750}\right)^2, \qquad s = 3.$$

Only by making the factor $\epsilon(n)\text{Max.}\{kL,1\}$ small can we satisfy these relations. Let kL exceed or equal unity - the least favorable situation. Then, <u>the conditions of Theorems (3) and (4) will be satisifed for all</u> $nr \in I_{\rho_o}$ <u>if the product of the number of wavelengths in the scattering region and the ϵ-error $\epsilon(n)$ is sufficiently small</u>. The content of this statement is best appreciated by considering a special case. In Example (1) of Section 2 we found that $W^n(x)$ approximated the uniform d.f. $W(x) = x$ with the error $\epsilon(n) = 1/n(j)$ at the j-th refinement. Within this context the underlined statement may be reexpressed as follows: The conditions of Theorems (3) and (4) will be satisfied if $L/\lambda n(j)$ is sufficiently small - that is, if the number of scatterers in boxes of wavelength-size, is, on the average, sufficiently large. This result is to be expected on intuitive grounds and in fact provides a justification for the formal method

sketched earlier when we discussed Born's treatment of the scattering of electromagnetic waves by a gas of dipoles.

Case (3): $nr \in I_{\rho_o} = \left\{ nr \mid 0 \leq nr \leq \rho_o \right\}$, n small

Since n is small [see neighborhood of (3.7)], $\beta_n = 1$. Substituting this value for β_n in (3.4) - (3.6) we find that

$$R_n^p \leq \left[1 + \rho_o/2\right](\alpha_n(\rho_o))^{p+1}, \tag{3.23}$$

$$\tilde{R}_n^p \leq \left[1 + \rho_o/2\right](\alpha_n(\rho_o))^{p+1}, \tag{3.24}$$

$$A_n^p = \left[1 + \rho_o/2\right](\alpha_n(\rho_o))^{p+1} \exp(\rho_o), \tag{3.25}$$

where

$$\alpha_n(\rho_o) = \rho_o(1 + \rho_o/2)\exp \rho_o \operatorname{Max.}\left\{1,\rho_o\right\} . \tag{3.26}$$

When $\alpha_n(\rho_o) < 1$, A_n^p can be made arbitrarily small by choosing p large enough. Now it is easy to verify that

$$\alpha(\rho_o) < 1 \quad \text{when} \quad \rho_o < \bar{\rho}_o \cong 0.625. \tag{3.27}$$

This result leads directly to the analog of Theorem (3), namely:

Theorem (5). Let nr be any point of I_{ρ_o}. Then in order that the right-hand side of

$$u^n(x) = \sum_{\nu=0}^{\infty} (nz/2Z)^\nu \, T_n^\nu U^n(x) \tag{3.28}$$

converge uniformly to the left-hand side for all x in the interval $-\infty < x < \infty$, it is sufficient to require that

$$nr < \bar{\rho}_o \cong 0.625. \tag{3.29}$$

The precise analog of Theorem (4) can also be proved; but we shall not do so here. It suffices to say that the relative and absolute errors R_n^p, $\tilde{R}_n^p$, A_n^p can, for fixed p, be made smaller than any given amount simply by choosing ρ_o small enough.

The relations (3.24)-(3.25) imply that U^n belonging to an arbitrary $W(x)$ will yield a good approximation for u^n if ρ_0 is sufficiently small; for, the right-hand sides of (3.24)-(3.25) are independent of $\epsilon(n)$. Because it makes little sense, in general, to speak here of $W(x)$ approximating $W^n(x)$, (3) is, for us, the least interesting of the cases. In the sequel we shall for the most part concern ourselves with Cases (1) and (2).

Before closing this section, we should mention that Theorem (1) is based on only one of the two types of estimates that we have derived in Part III for the quantities $||T_n^\nu u^n||/||u^n||$ and $||T_n^\nu U^n||/||U^n||$. The estimate used in Theorem (1) - the T_n^ν-type estimate as we shall call it - is derived in subsection 6.3; the other type, hereafter referred to as the $\hat{T}_n^\nu$-type estimate, is derived in subsection 6.4. In the sequel, as here, we shall, for the most part, favor the T_n^ν-type estimate because*, when n is large and $\nu \geq 2$, it behaves like $O[(\epsilon(n))^{\nu/2}]$ while, under the same circumstances, the $\hat{T}_n^\nu$-type estimate behaves like $O[(\epsilon(n))^{\nu/4}]$. The $\hat{T}_n^\nu$-type estimate, however, has several points in its favor among which the following may be noted: i) It is much easier to derive. ii) In the important case $\nu = 1$ both estimates vary with $\epsilon(n)$ like $O(\sqrt{\epsilon(n)})$, the factors implicit in the O-sign being comparable. iii) Finally, the $\hat{T}_n^\nu$-type estimate is in some respects better suited to dealing with the statistical problem in that it leads to more explicit, if less accurate, results than the T_n^ν-type estimate (see Theorem (7), subsection 5.3, also Remarks (9) and (11) in subsection 6.4).

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* For a more detailed discussion of the points made here and below see Remarks (9) - (11), subsection 6.4 - see also subsection 5.3 where the $\hat{T}_n^\nu$-type estimates are employed.

PART II. THE STATISTICAL PROBLEM

4. Formulation of the problem

The single configuration problem involved distributions of scatterers with fixed but arbitrary positions in the scattering region. Here, the positions are assumed to be identically distributed random variables, $x_1 = x_1(\theta)$, $x_2 = x_2(\theta), \ldots, x_n = x_n(\theta)$, θ denoting an arbitrary point of some probability space. In this section we shall be concerned mainly with formulating a suitable statistical analog (to be called $\mathcal{P}(\theta)$) of the property $\mathcal{P}$ of Part I. In a sense to be made more precise later [see Theorem (6)] 'almost all' sequences, $x_q(\theta)$, $q = 1,2,\ldots$, satisfying our $\mathcal{P}(\theta)$ will also satisfy $\mathcal{P}$ of Part I. This result furnishes the connecting link between our formulations of the single and random configuration problems and makes it possible to apply, without essential change, many of the results of Part I to the statistical problem.

We begin by introducing the probability space $(\mathbb{H}, \mathcal{B}, P)$. Here, $\mathbb{H}$ is a space consisting of points θ; $\mathcal{B}$ is a countably additive collection of sets of $\mathbb{H}$ and P is a probability measure on sets of $\mathcal{B}$. Second, we assume that the position of the q-th scatterer is a random variable $x_q = x_q(\theta)$ on $(\mathbb{H}, \mathcal{B}, P)$ such that

$$(4.1) \qquad P\left[\left\{\theta \,\middle|\, x_q(\theta) \le x\right\}\right] \equiv W(x) = \int_0^x w(\xi)\,d\xi, \qquad q = 1,2,\ldots,$$

where $W(x)$ is a continuum d.f. and $w(\xi)$ is a distribution density function defined as in Part I [see (2.9) and (2.10)]. Observe that the right-hand side of (4.1) is independent of q, $q = 1,2,\ldots$, in accordance with the requirement that the x_q's be identically distributed. Third, we assume a sequence of configurations is given which may be described as follows: i) $\mathcal{C}^j(\theta)$, the j-th random configuration, is the set of n(j) scatterer positions $\{x_1(\theta), x_2(\theta), \ldots, x_{n(j)}(\theta)\}$ where $n(j) \ge j$; ii) $\mathcal{C}^j(\theta)$ is obtained from $\mathcal{C}^{j-1}(\theta)$ simply by adding n(j)-n(j-1) scatterer positions, the labeling of the original n(j-1) elements in $\mathcal{C}^{j-1}(\theta)$ ~~earlier.~~ being retained. These $\mathcal{C}^j(\theta)$'s are the random configurations mentioned ~~in the title~~ earlier. From the nature of these $\mathcal{C}^j(\theta)$'s it follows that the scatterer positions $x_q(\theta)$, $q = 1,\ldots,n(j)$ in $\mathcal{C}^j(\theta)$ may be viewed as the first n(j) elements of an infinite sequence

x_q, $q = 1,2,\ldots$, formed in the natural way - with the original labeling - from the elements of $\mathcal{C}^j(\Theta)$, $j = 1,2,\ldots$. Fourth and finally, we consider the sequence of d.f.'s

$$(4.2) \qquad W^{n(j)}(x;\theta) \equiv \frac{1}{n(j)} \sum_{q=1}^{n(j)} H\left[x-x_q(\theta)\right], \qquad j = 1,2,\ldots,$$

[cf. (2.7) and (2.11)] and assume that there exists a null sequence $\epsilon(j;\theta)$ $j = 1,2,\ldots$, where $\epsilon(j,\theta)$, $j = 1,2,\ldots$, is independent of x, such that for each x, the relation

$$(4.3) \qquad |W^{n(j)}(x;\theta) - W(x)| \leq \epsilon(j;\theta), \qquad j = 1,2,\ldots,$$

holds except perhaps for a set $E = E(x)$ of P-measure zero.

If a sequence of d.f.'s $W^{n(j)}(x;\theta)$, $j = 1,2,\ldots$, obeys all four of the above requirements, we shall say that it has the property $\mathcal{P}(\theta)$, $\theta \in Ⓗ - E(x)$. We shall also say that $W(x)$ satisfies the property $\mathcal{P}(\theta)$, $\theta \in Ⓗ - E(x)$ with respect to the sequence $W^{n(j)}(x,\theta)$ $j = 1,2,\ldots$, and that the sequences $x_q(\theta)$. $q = 1,2,\ldots$, and $\mathcal{C}^j(\theta)$, $j = 1,2,\ldots$, satisfy the property $\mathcal{P}(\theta)$, $\theta \in Ⓗ - E(x)$.

Some remarks are in order concerning our formulation of $\mathcal{P}(\theta)$. First note that the elements of the null sequence have been allowed to depend on θ. We could, of course, require that $\epsilon(j;\theta)$ $j = 1,2,\ldots$, be independent of θ; but this would entail the exclusion of a large class of random processes $x_q(\theta)$ $q = 1,2,\ldots$, which would normally be employed in connection with the statistical problem [e.g., see Example (4) below]. Next, let us observe that

$$(4.4) \qquad \int_{\theta \in Ⓗ} W^{n(j)}(x;\theta)dP(\theta) = W(x), \qquad j = 1,2,\ldots \; .$$

In light of this, (4.3) states that the random sequence $\left\{H\left[x-x_q(\theta)\right],\ q = 1,2,\ldots\right\}$ satisfies, for each x, $0 \leq x \leq L$, a 'law of large numbers'. Usually such laws hold everywhere except on a set of P-measure zero; This is our set $E(x)$. Finally, let us stress the fact that every sequence $W^{n(j)}(x;\theta)$ $j = 1,2,\ldots$, which satisfies $\mathcal{P}(\theta)$ $\theta \in Ⓗ - E(x)$ is a convergent sequence for each x, $0 \leq x \leq L$, provided $\theta \notin E(x)$*.

The following theorem will enable us to establish the connection between $\mathcal{P}(\theta)$ and $\mathcal{P}$ and ultimately between the random and single configuration problems.

* The converse is also true- it is an almost immediate consequence of Conclusion ii) of Theorem 6 below.

Here and hereafter, unless specifically mentioned otherwise, we shall limit the statements and proofs of all results to the case in which $\mathcal{C}^j(\theta)$ contains $j = n$ scatterers; no difficulty will be encountered in extending these statements and proofs to the case in which $\mathcal{C}^j(\theta)$ contains more than j scatterers.

Theorem (6). Let the sequence

$$(4.5) \qquad W^n(x;\theta) = \frac{1}{n}\sum_{q=1}^{n} H\left[x-x_q(\theta)\right], \qquad n = 1,2,\ldots,$$

satisfy the property $\mathcal{P}(\theta)$, $\theta \in \textcircled{H} - E(x)$. Then there exists a set $\hat{E}$ of P-measure zero which i) is independent of x and ii)* has the property that $W^n(x;\theta)$ converges uniformly to $W(x)$ for all x, $0 \leq x \leq L$, whenever θ is in the set $\textcircled{H} - \hat{E}$.

Proof. Let $Y = \{y_1, y_2, \ldots\}$ be a countable, everywhere-dense set in the scattering interval — I say. Define $\hat{E}$ as follows:

$$\hat{E} = \bigcup_{\nu=1}^{\infty} E(y_\nu),$$

since $\hat{E}$ is the union of a countable collection of sets of P-measure zero we have

$$P(\hat{E}) = 0.$$

Our object now is to prove that $\hat{E}$ is the sought-after set - specifically, our object is to show that for each $\theta \in \textcircled{H} - \hat{E}$, $\lim_{n\to\infty} W^n(x;\theta) = W(x)$ for all $x \in I$. For each $\theta \in \textcircled{H} - \hat{E}$ it is clearly sufficient to prove this result for all $x \in I - Y$. Suppose therefore x is a fixed point of I-Y and let ξ_1 and ξ_2 be any two points of Y such that

$$\xi_1 < x < \xi_2;$$

the distance of ξ_2 from ξ_1 will be specified more precisely later. Then, because $W^n(x;\theta)$ is non-decreasing, we have

$$(4.6) \qquad W^n(\xi_1;\theta) \leq W^n(x;\theta) \leq W^n(\xi_2;\theta), \qquad \theta \in \textcircled{H} - \hat{E}.$$

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* Conclusion ii)* explains our not requiring that $\epsilon(j;\theta)$ depend on x in the formulation of $\mathcal{P}(\theta)$ - the uniform convergence with respect to x is automatic.

Letting n become infinite, we find:

$$(4.7)\qquad \begin{aligned} &\limsup W^n(x;\theta) \leq W(\xi_2;\theta), && \theta \in Ⓗ - \hat{E},\\ &\liminf W^n(x;\theta) \geq W(\xi_1;\theta), && \theta \in Ⓗ - \hat{E}. \end{aligned}$$

In addition, because, W(x) is a non-decreasing function of x, we have:

$$(4.8)\qquad W(\xi_1) \leq W(x) \leq W(\xi_2).$$

Now choose $\xi_1 \in Y$ and $\xi_2 \in Y$ so close that

$$(4.9)\qquad W(\xi_2) - W(\xi_1) \leq \eta ,$$

η being any preassigned positive number; this can always be done because W(x) is continuous. Using (4.8) and (4.9), we conclude that

$$(4.10)\qquad \begin{aligned} &W(\xi_2) \leq W(x) + \eta,\\ &W(\xi_1) \geq W(x) - \eta. \end{aligned}$$

Employing (4.7) and these results we find:

$$\begin{aligned} &\limsup_{n\to\infty} W^n(x;\theta) \leq W(x) + \eta , && \theta \in Ⓗ - \hat{E},\\ &\liminf_{n\to\infty} W^n(x;\theta) \geq W(x) - \eta , && \theta \in Ⓗ - \hat{E}. \end{aligned}$$

It follows immediately that for each $\theta \in Ⓗ - \hat{E}$ $\lim_{n\to\infty} W^n(x;\theta) = W(x)$ for all $x \in I-Y$. This is the result we set out to prove.

That for each $\theta \in Ⓗ - \hat{E}$ the convergence is uniform for all $x \in I$ is an immediate consequence of Polya's Theorem [see Part I p. 7].

Henceforth, any proposition which is valid for all θ except for those belonging to the exceptional set $\hat{E}$ will be said to be valid for <u>all sequences modulo $\hat{E}$</u> or in abbreviated form <u>a.s.(mod.$\hat{E}$)</u>; the sequences in question are those obtained from $W^n(x;\theta)$, n = 1,2,... or alternatively from $x_q(\theta)$, q = 1, 2,..., by letting θ vary over the range $Ⓗ - \hat{E}$. Theorem (6) implies that a.s.(mod.$\hat{E}$) satisfying $\mathcal{P}(\theta)$ also satisfy $\mathcal{P}$ of Part I. It is now clear how we must define the solution of the continuum problem U^n. In analogy to our procedure in Part I, we replace $W^n(x;\theta)$ in the integral equation for u^n [see (2.17)] by the common limit W(x) of the set of sequences

$\{W^n(x;\theta),\ n = 1,2,\ldots | \theta \notin E\}$ and obtain the integral equation for the corresponding U^n, namely (2.19).

The following are examples of random processes satisfying the property $\mathscr{P}(\theta)$.

Example (3)

Let Ⓗ be the half open interval $0 \leq x < 1$, $\mathscr{B}$ the class of Lebesgue measurable sets of Ⓗ. Let x_q, $q = 1,2,\ldots$, $0 < x_q < 1$, $q = 1,2,\ldots$, be a sequence of points which have the property of becoming uniformly everywhere dense in the unit interval. For such sequences, instances of which have been given in Examples 1 and 2 of Part I, a null sequence $\epsilon(j)$, $j = 1,2,\ldots$, can be found such that

$$(4.11) \qquad | W^{n(j)}(x) - x | \leq \epsilon(j), \qquad 0 \leq x \leq 1, \qquad j = 1,2,\ldots \ .$$

Now define $x_q(\theta)$ as follows:

$$x_q(\theta) = (\theta + x_q) \text{ mod.} 1 \qquad q = 1,2,\ldots,$$

where $x_q(\theta)$ is the smallest number in the unit interval congruent to $(\theta + x_q)$. The $x_q(\theta)$'s are evidently obtained from the x_q's simply by 'translating' the latter points by an amount θ. Making use of this fact, of the definition of $W^{n(j)}(x;\theta)$ and of (4.11), we can then easily show that

$$(4.12) \qquad | W^{n(j)}(x;\theta) - x | \leq 3\epsilon(j), \qquad 0 \leq x,\theta \leq 1.$$

The process $x_q(\theta)$, $q = 1,2,\ldots$, clearly satisfies the requirements of $\mathscr{P}(\theta)$ and, in fact, the convergence is uniform for all (x,θ) in the unit square. Moreover, the convergence is an 'everywhere' convergence; no set of measure zero need be omitted. Note that if x_q, $q = 1,2,\ldots$, is the sequence constructed in Example (1) then

$$(4.13) \qquad | W^{n(j)}(x;\theta) - x | \leq 3/2^j$$

for configuration $C^j(\theta)$. Similarly, if x_q, $q = 1,2,\ldots$, is the sequence of Example (2) and γ is a quadratic irrational then since $n = j$

$$(4.14) \qquad | W^n(x,\theta) - x | \leq O(\log n/n),$$

where the factor 3 has been absorbed into the 'O'-sign.

Example (4)

Let $x_q(\theta)$, $q = 1,2,\ldots$, $0 \leq x_q(\theta) \leq 1$, $\theta \in Ⓗ$, be any strictly stationary, metrically indecomposable, random sequence on $(Ⓗ, \mathcal{B}, P)$ such that

$$P\left[\left\{\theta | x_q \leq x\right\}\right] = W(x) \qquad q = 1,2,\ldots \quad .$$

In virtue of this assumption, the sequence $H[x-x_q(\theta)]$ $q = 1,2,\ldots$, is, for each x, also a strictly stationary, metrically indecomposable, random sequence [see [9]. p. 458 for a discussion of this point] and it follows from the ergodic property of such sequences [cf. [9], p. 465] that

$$\lim_{n\to\infty} \left\{\left(\sum_{q=1}^{n} H\left[x-x_q(\theta)\right]\right) \Big/ n\right\} = \lim_{n\to\infty} W^n(x;\theta) = \int_{\theta \in Ⓗ} H\left[x-x_1(\theta)\right] dP(\theta)$$

$$= W(x)$$

for all $\theta \in Ⓗ$ except for a set $E(x)$ of P-measure zero. A concrete example of a process of the type under consideration is the process [see Example (3)] $x_q(\theta) = \left[\theta + (q-1)\gamma\right]$ mod.1, $q = 1,2,\ldots$, and γ irrational.

Example (4a)

Perhaps the simplest example of a process satisfying the requirements of Example (4) is that of a sequence of independent random variables. In this case it follows from the 'law of iterated logarithm', (cf. [10]p. 61) for each x, and all θ with the exception of a set $E(x)$ of P-measure zero, that

$$|W^n(x;\theta) - W(x)| \leq (1+ \epsilon_1)\sigma \left[\log\left(\log n(\theta)\right)/n(\theta)\right]^{1/2}$$

for a suitably large n which depends in general on θ. Here ϵ_1 is any(arbitrarily small) positive number and

$$\sigma = \left[\int_{\theta \in Ⓗ} \left\{H\left[x-x_1(\theta)\right] - W(x)\right\}^2 dP(\theta)\right]^{1/2} = \left[W(x)\left(1 - W(x)\right)\right]^{1/2} \leq 1.$$

Thus, we get finally,

$$(4.15) \qquad |W^n(x;\theta) - W(x)| \leq (1 + \epsilon_1)\left[\log\left(\log n(\theta)\right) \Big/ n(\theta)\right]^{1/2}.$$

5. Results for the statistical problem

5.1 Preliminary remarks - Notation. Throughout this section it will be assumed that the sequence $W^n(x;\theta)$, $n = 1,2,\ldots$, satisfies $\mathscr{P}(\theta)$, $\theta \in$ Ⓗ - $\hat{E}$ - i.e., that a.s.(mod.$\hat{E}$) satisfy $\mathscr{P}(\theta)$.

Our discussion will be subdivided into three parts. In subsection (5.2) we shall discuss the special case in which the terms of the null sequence $\epsilon(n;\theta)$, $n = 1,2,\ldots$, are assumed to be independent of θ; Example(3) shows that such null sequences exist. In subsection (5.3) we shall discuss the general case, in which the $\epsilon(n;\theta)$ are permitted to depend on θ. For the sake of brevity, we shall concern ourselves with only one of three types of errors treated earlier in Part I, namely, the absolute error,

$$A_n^p(\theta) = \| u^n(x;\theta) - \sum_{\nu=1}^{p} (nz/2Z)^\nu \; T_n^\nu U^n(x) \| , \tag{5.1}$$

where by $u^n(x;\theta)$ we mean*

$$u^n(x;\theta) \equiv u^n(x;x_1(\theta),x_2(\theta),\ldots,x_n(\theta)). \tag{5.2}$$

Since a.s.(mod.$\hat{E}$) satisfying $\mathscr{P}(\theta)$, also satisfy $\mathscr{P}$, it follows from Theorem (1) that

$$A_n^p(\theta) \leq \left[1 + (nr/2)\right] \exp(nr)\left(\alpha_n \beta_n(\theta)\right)^{p+1}, \qquad \text{a.s.(mod.E)}, \tag{5.3}$$

where α_n and β_n are defined as in (3.2) and (3.3). In addition, we shall consider the averages

$$(A_n^p)_{av} = \| < u^n > - \sum_{\nu=1}^{p} (nz/2Z)^\nu < T_n^\nu U^n > \| , \tag{5.4}$$

and

$$(A_n^p)_{rms} = < \left(A_n^p(\theta)\right)^2 >^{1/2} . \tag{5.5}$$

Here, if $f(\theta)$ is a random variable on the probability space (Ⓗ, $\mathscr{B}$, P), then

$$< f > \equiv < f(\theta) > \equiv \int_{\theta \in Ⓗ} f(\theta) dP(\theta) . \tag{5.6}$$

Using Theorem (1), we can show that

---

* Recall that $U^n(x)$ is independent of θ.

$$(5.7) \qquad (A_n^p)_{av} \leq \left[1 + (nr/2)\right] \exp(nr)(\alpha_n)^{p+1} < (\beta_n(\theta))^{p+1} >$$

$$(5.8) \qquad (A_n^p)_{rms} \leq \left[1 + (nr/2)\right] \exp(nr)(\alpha_n)^{p+1} < (\beta_n(\theta))^{2p+2} >^{1/2} ,$$

where

$$(5.9) \qquad < \left(\beta_n(\theta)\right)^{p+1} > \leq \left[6\sqrt{2\text{Max.}\{1,kL\}} \left\langle \left(\epsilon(n;\theta)\right)^{\frac{p+1}{2}} \right\rangle^{\frac{1}{p+1}} \right]^{p+1} ,$$

$$(5.10) \qquad < \left(\beta_n(\theta)\right)^{2p+2} >^{1/2} \leq \left[6\sqrt{2\text{Max.}\{1,kL\}} \left\langle \left(\epsilon(n;\theta)\right)^{p+1} \right\rangle^{\frac{1}{2p+2}} \right]^{p+1} .$$

The relation (5.8) follows directly from (5.5) and (3.6). In deriving (5.7), we have made use of (5.4) and the relation

$$\left\| \left\langle D_n^p(x;\theta) \right\rangle \right\| \leq \left\langle \left\| D_n^p(x;\theta) \right\| \right\rangle , \qquad p = 0,1,2,\ldots$$

$$\left[D_n^p(x;\theta) \equiv u^n(x;\theta) - \sum_{\nu=1}^{p} (nz/2Z)^\nu T_n^\nu U^n(x) \right] ,$$

which in turn is a consequence of the inequalities

$$\underset{-\infty < x < \infty}{\text{Maximum}} \left| \left\langle D_n^p(x;\theta) \right\rangle \right| \leq \underset{-\infty < x < \infty}{\text{Maximum}} \left\langle \left| D_n^p(x;\theta) \right| \right\rangle \leq \left\langle \underset{-\infty < x < \infty}{\text{Maximum}} \left| D_n^p(x;\theta) \right| \right\rangle .$$

The inequalities (5.9) and (5.10) follow directly from the definition of β_n [see (3.3)]. Actually the sharper estimates

$$< \left(\beta_n(\theta)\right)^{p+1} > \leq \left[\text{Min.} \left\{ 1, 6\sqrt{2\text{Max.}\{1,kL\}} \left\langle \left(\epsilon(n;\theta)\right)^{\frac{p+1}{2}} \right\rangle^{p+1} \right\} \right]^{p+1} ,$$

$$< \left(\beta_n(\theta)\right)^{2p+2} > \leq \left[\text{Min.} \left\{ 1, 6\sqrt{2\text{Max.}\{1,kL\}} \left\langle \left(\epsilon(n;\theta)\right)^{p+1} \right\rangle^{\frac{1}{2p+2}} \right\} \right]^{p+1}$$

are valid; however, for the sake of simplicity, we shall use only (5.9) and (5.10) in the sequel.

Our object in subsection (5.2) and (5.3) will be to derive the various statistical analogs of Theorems (2) - (4) of Part I - specifically to give sufficient conditions which will guarantee the smallness of A_n^p, $(A_n^p)_{av}$ and $(A_n^p)_{rms}$. In the last subsection (5.4) we shall sketch Foldy's formal procedure and show how the results of this procedure may be justified on the basis of the results of subsections (5.2) and (5.3). We shall also calculate explicitly the first terms of the series expressions for $< u^n >$ and $< |u^n|^2 >$ for several types of random processes.

5.2 Null sequence $\epsilon(n;\theta)$, $n = 1,2,\ldots$, independent of θ. In this case $\beta_n(\theta)$ is independent of θ; consequently the right-hand sides of (5.7) and (5.8) reduce to the right-hand side of (5.3) with θ absent. From this follows that if A_n^p is replaced by $(A_n^p)_{av}$ or $(A_n^p)_{rms}$ in Theorems (3) and (4) of Part I, then the resulting statements are valid and apply without further change to the present problem. This is of course to be expected; for, if $A_n^p(\theta)$ is substituted for A_n^p in the statement of these theorems the results again apply without change to the present problem with however, one proviso: the final statements must be augmented by the phrase "a.s. (mod. E)" in suitable places. Similar remarks apply to the other theorems of Part I.

For example, Theorems (2) and (3) become:

Theorem (2s)*. Let $r = \rho_o'/n$ where $\rho_o' > 0$. Then the $\lim_{n\to\infty} u^n(x,\theta)$ exists for a.s.(Mod.E) and

$$\lim_{n\to\infty} ||u^n(x;\theta) - U(x)|| = 0 \quad , \qquad \text{a.s.(Mod. E)} , \tag{5.11}$$

where U is independent of n and θ and satisfies the integral equation (2.19).

Theorem (3s). Let nr be any point of I_{ρ_o} $\left[= \left\{ nr | 0 \le nr \le \rho_o \right\} \right]$. Then in order that the right-hand side of

$$u^n(x;\theta) = \sum_{\nu=0}^{\infty} (nz/2Z)^{\nu} T_n^{\nu} U^n(x) \tag{5.12}$$

may converge to the left-hand side uniformly in x, $-\infty < x < \infty$, for a.s. (Mod. E) it is sufficient to require that kL, $\epsilon(n)$ and nr satisfy the following relation for a.s.(Mod.E):

$$(nr)^2 \epsilon(n) \text{ Max.}\left\{kL, 1\right\} \le \frac{1}{2} \left[\frac{\rho_o}{6\alpha_n(\rho_o)}\right]^2 \quad . \tag{5.13}$$

---

* Read as 'Theorem 2, statistical'.

5.3 Null sequence $\epsilon(n;\theta)$, $n = 1,2,..$, dependent on θ, $\theta \in \textcircled{H} - \hat{E}$. It is easy to verify merely by inspecting the proof of Theorem (2) (Part I) that its analog Theorem (2s) applies as well to the present case as to the previous case. We may therefore limit ourselves to discussing finite values of n. Here, the precise equivalence between the single and the random configuration problem breaks down. For, in general, if $\epsilon(n;\theta)$ depends on θ the possibility arises that there exists for a given n a set of non-zero P-measure on which the conditions of Theorems (3) and (4) are not met. Therefore, apart from Theorem (2s), we must give up hope of obtaining theorems which hold for a.s.(Mod.E). The possibility however remains that the quantities $\langle(\epsilon(n;\theta))^{(p+1)/2}\rangle$ and $\langle(\epsilon(n;\theta))^{p+1}\rangle$ can be made small in which case $(A_n^p)_{av}$ and $(A_n^p)_{rms}$ can also be made small. Beginning with (5.7) and (5.8) it is clear how to formulate the analogs of Theorem (4): if in Theorem (4) the quantities in column (a) of the table below are replaced by the corresponding quantities in columns (b) and (c), then the resulting statements are valid and apply without further change to the present problem.

(a)	(b)	(c)
A_n^p	$(A_n^p)_{av}$	$(A_n^p)_{rms}$
$\epsilon(n)$	$\langle(\epsilon(n;\theta))^{(p+1)/2}\rangle$	$\langle(\epsilon(n;\theta))^{p+1}\rangle^{1/2}$

In addition, we have the following analog of Theorem (3):

Theorem (3's). In order that the right-hand side of

$$(5.14) \qquad u^n(x;\theta) = \sum_{\nu=0}^{\infty} (nz/2Z)^{\nu} \langle T_n^{\nu} U^n \rangle$$

may converge to the left-hand side uniformly in x, $-\infty < x < \infty$, and $nr \in I_{\rho_0} = \{nr \,|\, 0 \leq nr \leq \rho_0, \rho_0 > 0\}$ it is sufficient to require that $kL, \epsilon(n)$ and nr satisfy the following relation:*

$$(5.15) \qquad (nr)^2 \cdot \limsup_{p\to\infty}\left[\langle(\epsilon(n;\theta))^{(p+1)/2}\rangle^{1/p+1}\right] \mathrm{Max.}\{kL, 1\} \leq \frac{1}{2}\left(\frac{\rho_0}{6\alpha_n(\rho_0)}\right)^2 .$$

---

* If $\langle(\epsilon(n;\theta))^{(p+1)/2}\rangle^{1/p+1}$ is replaced by $\langle(\epsilon(n;\theta))^{p+1}\rangle^{1/(2p+2)}$ the resulting relation is a sufficient condition for the (5.14) to hold in the root-mean-square sense.

Under certain circumstances, when p = 0, the results for $(A_n^p)_{av}$ mentioned above (see Table) may be somewhat improved. Since this case is an important one, we shall treat it in detail. We begin with the observation that

$$(5.16) \qquad < T_n U^n > \;=\; \left\langle \int_0^L (2ik) G^n(x,x') U^n(x') d\left[W^n(x;\theta) - W(x)\right] \right\rangle \;=\; 0,$$

[see (2.22) and (4.4)]. From (5.4) it follows that

$$(5.17) \qquad (A_n^1)_{av} \;=\; \| < u^n > - \left[U^n + (nz/2Z) < T_n U^n >\right] \| \;=\; \| < u^n > - U^n \| .$$

Setting p = 1 in (5.7) we have, in virtue of (5.9)

$$(5.18) \qquad \frac{\| < u^n > - U^n \|}{[1 + (nr/2)]} \leq \left[6\alpha_n \sqrt{2\text{Max.}\left\{kL, 1\right\} \;\; < \epsilon(n;\theta) >^{1/2}} \right]^2 .$$

This result may be compared with

$$(5.19) \qquad \frac{\| < u^n > - U^n \|}{[1 + (nr/2)]} \leq 6\alpha_n \sqrt{2\text{Max}\{kL, 1\} < \left(\epsilon(n;\theta)\right)^{1/2} >}$$

which is obtained from (5.10) by setting p = 0. Since $< \left(\epsilon(n;\theta)\right)^{1/2} >$ regarded as a function of n is, roughly speaking, the same order of magnitude as $< \epsilon\,(n;\theta) >^{1/2}$ * the terms within the square root signs are comparable. Evidently, then, the estimate in (5.18) is an improvement on that given in (5.19) when i) nr is such that $\alpha_n < 1$ and ii) the factor $6\sqrt{2\text{Max}\left\{kL, 1\right\} < \epsilon(n;\theta) >^{1/2}}$ is sufficiently small compared with unity.

Employing (5.18) we obtain the following

<u>Theorem (4s)</u> Let $nr \in I_{\rho_o} = \left\{ nr \,|\, 0 \leq nr \leq \rho_o,\; \rho_o > 0 \right\}$ and let δ be any positive number no matter how small. Then in order that the relation

$$(5.20) \qquad \| < u^n(x;\theta) > \; - \; U^n \| \;\leq\; \delta$$

may hold, it is sufficient to require that

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* In fact, when $\epsilon(n;\theta)$ is independent of θ, these quantities are equal.

$$(5.21) \qquad (nr)^2 \mathrm{Max.}\left\{kL, 1\right\} < \epsilon(n;\theta) > \;\leq\; \frac{\rho_o^2\,\delta}{72\alpha_n(\rho_o)^2\left[1+(nr/2)\right]} \;.$$

We wish to illustrate another procedure for determining conditions which will ensure the smallness of $(A_n^p)_{av}$ and $(A_n^p)_{rms}$. We shall do this by estimating the magnitude of

$$(5.22) \qquad (A_n^o)_{av} = (A_n)_{av}.$$

In this new procedure the $\hat{T}_n$-types estimates, mentioned at the end of Section 3, and the quantity

$$(5.23) \qquad \epsilon(n;\theta,x) \equiv |W^n(x;\theta) - W(x)|$$

replace the T_n^ν-types estimates and $\epsilon(n;\theta)$ respectively in our earlier discussion. The virtue of the present method as compared with our earlier one is this: The quantity $<(\epsilon(n;\theta,x))^2>$, which appears in the final estimates, may be calculated explicitly as a function of n and the range of correlation (to be defined below) of the underlying set of random variables $x_1(\theta),\ldots,x_n(\theta)$, whereas the earlier discussion depended upon our knowing the less fundamental quantity $\epsilon(n;\theta) > |W^n(x,\theta) - W(x)|$. On the other hand, it must be said that the new procedure appears, in several respects, to give less accurate results than the older one.

We begin with the following observation:

$$(5.24) \qquad (A_n)_{av} = \left\|\left\langle u^n(x;\theta)\right\rangle - U^n\right\| = \frac{n|z|}{2Z}\left\|< T_n u^n >\right\| \leq (nr/2)\left\langle \| T_n u^n \|\right\rangle .$$

The problem of estimating $(A_n)_{av}$ therefore reduces to that of estimating $\left\langle \|T_n u^n\|\right\rangle$. To this end, let us define the quantities $\hat{\alpha}_n$ and $\hat{\beta}_n$ as follows:

$$(5.25) \qquad \hat{\alpha}_n = (nr)\exp(nr)\left[1 + (nr/2)\exp(nr)\right] ,$$

$$(5.26) \qquad \hat{\beta}_n(\theta) = \mathrm{Max.}\left\{1, \left[\frac{2kL}{m} + \frac{1}{2}\bullet\sum_{j=o}^{m-1}\left|\int_{\xi_j}^{\xi_{j+1}} d\left[W^n(x;\theta)-W(x)\right]\right|\right]\right\}.$$

Here, m is an arbitrary positive integer and $0 = \xi_o < \xi_1 < \ldots < \xi_m = L$ are $m + 1$ points that subdivide the interval $0 \leq x \leq L$ into m subintervals of equal length. Employing (6.103)-(6.105), (6.58), (6.1), (6.24) of Part III and (5.24)-(5.26) we find:

$$\| u^n(x;\theta)-U^n \| \leq \hat{\alpha}_n(nr)\exp(nr)< \hat{\beta}_n(\theta) >. \tag{5.27}$$

From the definition of $\hat{\beta}_n(\theta)$ and $\epsilon(n;\theta,x)$ it follows that

$$< \hat{\beta}_n(\theta) > \leq \text{Min.} \left\{1, \left[\frac{2kL}{m} + m \underset{0\leq x \leq 1}{\text{l.u.b}} < \epsilon(n;\theta,x) >\right]\right\}$$

and hence that

$$< \hat{\beta}_n(\theta) > \leq \text{Min.} \left\{ 1, \left[\frac{2kL}{m} + m \underset{0\leq x \leq 1}{\text{l.u.b}} < \epsilon(n;\theta,x)^2 >^{1/2}\right] \right\}, \tag{5.28}$$

the last relation being a consequence of Schwartz's inequality. Let B(m) denote the term in square brackets, m_o the value at which B(m) assumes its minimum, and m_o' the smallest integer equal to or exceeding m_o. Then, because the relation (5.28) is valid for all $m \geq 1$ it follows that

$$< \hat{\beta}_n(\theta) > \leq \text{Min.} \left\{1, B(m_o')\right\} . \tag{5.29}$$

Now it is easy to verify that

$$m_o = \sqrt{2kL/(\epsilon_n)_{av}} ,$$

where we have set

$$(\epsilon_n)_{av} = \underset{0\leq x \leq L}{\text{l.u.b}} < \epsilon(n;\theta,x)^2 >^{1/2} . \tag{5.30}$$

It follows that

$$B(m_o') = 2kL/m_o' + m_o'(\epsilon_n)_{av} \leq (2kL/m_o) + (m_o+1)(\epsilon_n)_{av}$$

$$\leq 2 \sqrt{2kL(\epsilon_n)_{av}} + (\epsilon_n)_{av}$$

and hence that*

$$B(m_o') \leq 3 \sqrt{\text{Max.} \left\{1,kL\right\} (\epsilon_n)_{av}} . \tag{5.31}$$

---

* Since $\epsilon(n;\theta,x)$ is at most equal to unity, the same is true of $(\epsilon_n)_{av}$ (see (5.30)).

Employing (5.27), (5.29) and (5.31), we find that

$$(5.32) \qquad \| < u^n(x;\theta) > - U^n \| \leq \hat{a}_n \exp(nr) \mathrm{Min.} \left\{ 1, 3\sqrt{2(\epsilon_n)_{av} \mathrm{Max.} \{kL, 1\}} \right\}.$$

This estimate is similar in form to the estimates for $(A_n)_{av}$ discussed earlier but with an essential difference. Here, $(\epsilon_n)_{av}$ involves the average of $\left(W^n(x;\theta)-W(x)\right)^2$ which may be estimated explicitly in terms of n.

We turn now to the problem of estimating $(\epsilon_n)_{av}$. We begin by defining the 'range of correlation' of a stochastic process $x_q(\theta)$, $q = 1,2,\ldots$. The integer b ($b \geq 0$) is said to be the range of correlation of a process $x_q(\theta)$, $q = 1,2,\ldots$, if x_q and x_{q+p} are independent when and only when $p>b$. We shall assume henceforth that $x_q(\theta)$ $q = 1,2,\ldots$, is strictly stationary. Combining (5.30), (4.2) and (2.7), and averaging, we easily verify that*

$$(5.33) \qquad (\epsilon_n)_{av} = (1/n)^{1/2} \underset{0 \leq x \leq 1}{\mathrm{l.u.b}} \left\{ \left[W(x)-(W(x))^2 \right] + \frac{2(n-1)}{n} \left[W^{1,2}(x)-(W(x))^2 \right] + \ldots + \frac{2(n-b)}{n} \left[W^{1,b}(x,x)-(W(x))^2 \right] \right\}$$

$$\leq (2/n)^{1/2} \left\{ n/2 + (n-1) + (n-2) + \ldots + (n-b) \right\}^{1/2}$$

or

---

* Note that .

$$(\dagger) \qquad < (W^n(x;\theta) - W(x))^2 > = < \left(W^n(x)\right)^2 > - \left(W(x)\right)^2$$

and

$$(\dagger\,\dagger) \qquad n^2 < (W^n(x))^2 > = \left\langle \left(\sum_{q=0}^{n} H^2\left(x-x_q(\theta)\right) \right) \right\rangle + \left\langle \left({\sum_{q,s=1}^{n}}' H\left(x-x_q(\theta)\right) H\left(x-x_s(\theta)\right) \right) \right\rangle,$$

where the prime in the second expression on the right indicates the omission of the term s = q. From ($\dagger\dagger$) and the definition of H(x), [see (2.7)], it follows that

$$n^2 < \left(W^n(x)\right)^2 > = nW(x) + 2(n-1)W^{1,2}(x,x) + \ldots + 2(n-b)W^{1,b}(x,x) + \left\{ n^2 - \left[n + 2(n-1) + \ldots + 2(n-b) \right] \right\} \left(W(x)\right)^2.$$

This relation and ($\dagger$) lead directly to (5.33).

$$(5.34)\qquad (\epsilon_n)_{av} \leq \sqrt{\frac{1+2b}{n}}\left[1-\frac{b(b+1)}{(1+2b)n}\right]^{1/2} \leq \sqrt{\frac{1+2b}{n}}\ ;$$

here, $W^{1,j}(x_1,x_j)$ is the joint probability d.f. of $x_1(\theta)$ and $x_j(\theta)$. Combining (5.34) and (5.32) we finally find that

$$(5.35)\qquad (A_n)_{av} = \exp(nr)\hat{\alpha}_n(nr)\mathrm{Min.}\left\{1,\ 3\sqrt{2\left(\frac{1+2b}{n}\right)^{1/2}\mathrm{Max.}\left\{kL,1\right\}}\right\}\ .$$

Our results may be summarized as follows:

<u>Theorem (7)</u>. Let nr be any point of $I_{\rho_o} \equiv \left\{nr \mid 0 \leq nr \leq \rho_o,\ \rho_o > 0\right\}$, and let δ be any preassigned positive number. Then in order that

$$(A_n)_{av} \leq \delta$$

may hold, the following relation must be satisfied:

$$(5.36)\qquad nr\ \mathrm{Max.}\left\{1, 3\sqrt{2\left(\frac{1+2b}{n}\right)^{1/2}\mathrm{Max}\left\{kL,1\right\}}\right\} \leq \frac{\rho_o\delta}{\exp(\rho_o)\alpha_n(\rho_o)}\ .$$

This relation can always be satisfied by taking ρ_o small enough. Of greater interest for us is the fact that for fixed ρ_o (5.36) can be satisfied for all $nr \in I_{\rho_o}$ provided i) the range of correlation b is finite and ii) $b \ll n$.

5.4 <u>A justification of the Foldy method - other remarks</u>. Foldy's[5] method, as applied to one-dimensional problems, begins with the equation

$$(5.37)\qquad u^n\left[x;x_1(\theta),\ldots,x_n(\theta)\right] = \exp(ikx)-(z/2Z)\sum_{q=1}^{n}\exp(ik|x-x_q(\theta)|)\cdot u^n\left[x_q(\theta);\ x_1(\theta),\ldots,x_n(\theta)\right]$$

which is simply another form of the integral equation (2.17). Let us assume, as does Foldy, that the scatterer positions are identically distributed, mutually independent, random variables and let

$$(5.38)\qquad w(x_1,x_2,\ldots,x_n) = w(x_1)w(x_2)\ldots w(x_n)$$

be the joint probability density function of these random variables. Averaging both sides of (5.37) and taking into account the fact that $u^n(x;\theta)$ is invariant under any permutation of the scatterer positions, we find that

$$(5.39)\qquad < u^n(x) > = \exp(ikx) - (nz/2Z) \int_0^L \exp(ik|x-x'|) < u^n(x') >^{x_1} w(x')dx',$$

where

$$(5.40)\qquad < u^n(x) >^{x_1} \equiv \overbrace{\int_0^L \cdots \int_0^L}^{n-1} u^n(x;x_1,x_2,x_3,\ldots,x_n)\left(\prod_{j=2}^{n-1} w(x_j)wdx_2dx_3\ldots dx_n\right) .$$

The fundamental approximation of the Foldy method is based on the belief that, when n is sufficiently large, the function u^n averaged over all scatterer positions is approximately equal to u^n averaged over all scatterer positions but one- specifically, that

$$(5.41)\qquad < u^n(x) > \cong < u^n(x) >^{x_1}.$$

It arises out of the desire to make (5.39) into a genuine integral equation; for, if we accept (5.41), equation (5.39) becomes

$$(5.42)\qquad < u^n(x) >^{x_1} \cong \exp(ikx) - (nz/2Z) \int_0^L \exp(ik|x-x'|)< u^n(x') >^{x_1} w(x')dx'$$

which is simply the (approximate) integral equation for U^n [cf. (2.19)]. Thus the fundamental approximation of the Foldy method has the effect of identifying $< u^n >$ with U^n. This is evidently justifiable only when

$$(A_n)_{av} = ||< u^n(x) > - U^n||$$

is sufficiently small. Here, we are in a position to use our earlier results. In fact, Theorems* (4s) and (7) are immediately applicable and furnish sufficient conditions which guarantee the smallness of $(A_n)_{av}$.

In the following we shall derive approximate expressions for $< u^n >$ and $< |u^n|^2 >$. Our aim will be to express all averages in terms of the d.f.s. (individual and joint) of the underlying random process $x_q(\theta)$, $q = 1,2,\ldots$ For the sake of simplicity we shall carry out our calculations explicitly only for terms of order two or less in the variable $n|z|/2Z$; it is in this sense that our results will be approximate. Even with this limitation, the procedure for obtaining results in the desired form can become involved. The following is an outline of a procedure that appears to be the most efficient.

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* See pages 31 and 35.

Taking averages of $u^n(x;\theta)$ and $|u^n(x;\theta)|^2$ we find:

(5.43) $$< u^n > = U^n + (nz/2Z)^2 < (x)T_n^2U^n > + \ldots \quad ,$$

(5.44) $$< |u^n|^2 > = U^n + (n|z|/2Z)^2\left\{ < |(x)T_nU^n|^2 > - 2\mathrm{Re}\; U^n < T_n^2U^n > \right\} + \ldots +$$

since , as we showed earlier [see (5.16)],*

$$< (x)T_nU^n > \;=\; < \overline{(x)T_nU^n} > \;=\; 0.$$

The simplest way of evaluating the averages of $(x)T_n^2U^n$ and $|(x)T_nU^n|^2$ is to express the integrals involving $dW^n(x)$ as finite sums and to average term-wise. The final results of this procedure are **:

(5.45) $$\begin{aligned} < T_n^2U^n(x)> = \frac{(2ik)^2}{n}\int_0^L G^n(x,\xi)G^n(\xi,\xi)U^n(\xi)w(\xi)d\xi \\ + \frac{(2ik)^2}{n^2}\sum_{j,m=1}^{n}{}' \int_0^L\int_0^L G^n(x,\xi_j)G^n(\xi_j,\xi_m)U^n(\xi_m)w^{j,m}(\xi_j,\xi_m)d\xi_1 d\xi_m \\ - (2ik)^2\int_0^L\int_0^L G^n(x,\xi_1)G^n(\xi_1,\xi_2)U^n(\xi_2)w(\xi_1)w(\xi_2)d\xi_1 d\xi_2 \end{aligned}$$

(5.46) $$\begin{aligned} <|T_nU^n(x)|^2> = \frac{2k^2}{n}\int_0^L |G^n(x,\xi)|^2|U^n(\xi)|^2 w(\xi)d\xi \\ + \frac{2k^2}{n^2}\sum_{j,m=1}^{n}\int_0^L G^n(x,\xi_j)\overline{G^n(x,\xi_m)}U^n(\xi_j)\overline{U^n(\xi_m)}w^{j,m}(\xi_j,\xi_m)d\xi_j d\xi_m \\ - (2k)^2\left|\int_0^L G^n(x,\xi)U^n(\xi)w(\xi)d\xi\right|^2 \end{aligned}$$

where $w^{j,m}(\xi_j,\xi_m)$ is the joint probability density of the random variables $x_j(\theta)$ and $x_m(\theta)$. The prime on the summation sign indicates the omission of all 'diagonal' terms, i.e., the first terms in (5.45) and (5.46).

---

* $\overline{Q}$ denotes the complex conjugate of Q.

** See pp. 10 and 11 for the definitions of T_n^2 and G^n, respectively.

If the underlying random process $x_q(\theta)$ $q = 1,2,\ldots,$ is a sequence of independent random variables, we find that

$$(5.47) \qquad < T_n^2 U^n(x) > = \frac{(2ik)^2}{n} \left\{ \int_0^L G^n(x,\xi) G^n(\xi,\xi) U^n(\xi) w(\xi) d\xi - \int_0^L G^n(x,\xi_1) G^n(\xi_1,\xi_2) U^n(\xi_2) w(\xi_1) w(\xi_2) d\xi_1 d\xi_2 \right\} .$$

$$(5.48) \qquad < |TU^n(x)|^2 > = \frac{2k^2}{n} \left\{ \int_0^L |G^n(x,\xi)|^2 |U^n(\xi)|^2 w(\xi) d\xi - \left| \int_0^1 G^n(x,\xi) U^n(\xi) w(\xi) d\xi \right|^2 \right\} .$$

More generally, if $x_q(\theta)$ $q = 1,2,\ldots,$ is a strictly stationary random sequence having the range of correlation b, then (5.45) and (5.46) become:

$$< T_n^2 U^n(x) > = \frac{(2ik)^2}{n^2} \left\{ n \int_0^L G^n(x,\xi) G^n(\xi,\xi) U^n(\xi) w(\xi) d\xi - n \int_0^L \int_0^L G^n(x,\xi_1) G^n(\xi_1,\xi_2) \cdot U^n(\xi_2) w(\xi_1) w(\xi_2) d\xi_1 d\xi_2 \right.$$

$$(5.49) \qquad + 2(n-1) \int_0^L \int_0^L G^n(x,\xi_1) G^n(\xi_1,\xi_2) U^n(\xi_2) \left[w^{1,2}(\xi_1,\xi_2) - w(\xi_1) w(\xi_2) \right] d\xi_1 d\xi_2$$

$$+ 2(n-2) \int_0^L \int_0^L G^n(x,\xi_1) G^n(\xi_1,\xi_2) U^n(\xi_2) \left[w^{1,3}(\xi_1,\xi_2) - w(\xi_1) w(\xi_2) \right] d\xi_1 d\xi_2$$

$$+ \\ \vdots$$

$$\left. + \left[2(n-b) \right] \int_0^L \int_0^L G^n(x,\xi_1) G^n(\xi_1,\xi_2) U^n(\xi_2) \left[w^{1,b}(\xi_1,\xi_2) - w(\xi_1) w(\xi_2) \right] d\xi_1 d\xi_2 \right\},$$

$$\langle |(x)T_nU^n|^2\rangle = \frac{2k^2}{n^2}\Bigg\{ n\int_0^L |G^n(x,\xi)|^2|U^n(\xi_2)|^2 w(\xi)d\xi - n\left|\int_0^L G^n(x,\xi)U^n(\xi)w(\xi)d\xi\right|^2$$

$$+ 2(n-1)\int_0^L\int_0^L G^n(x,\xi_1)G^n(x,\xi_2)U^n(\xi_1)\overline{U^n}(\xi_2)\left[w^{1,2}(\xi_1,\xi_2)-w(\xi_1)w(\xi_2)\right]d\xi_1 d\xi_2$$

$$+ 2(n-2)\int_0^L\int_0^L G^n(x,\xi_1)\overline{G^n(x,\xi_2)}U^n(\xi_1)\overline{U^n}(\xi_2)\left[w^{1,3}(\xi_1,\xi_2)-w(\xi_1)w(\xi_2)\right]d\xi_1 d\xi_2 \tag{5.50}$$

$$+ \\ \vdots$$

$$+ \boxed{2(n-b)}\int_0^L\int_0^L G^n(x,\xi_1)\overline{G^n(x,\xi_2)}U^n(\xi_1)\overline{U^n}(\xi_2)\left[w^{1,b}(\xi_1,\xi_2)-w(\xi_1)w(\xi_2)\right]d\xi_1 d\xi_2\Bigg\}.$$

If the range of correlation is infinite then b must be set equal to n-1 in (5.50).

When

$$x_q(\theta) = \left[\theta + (q-1)\gamma\right] \bmod. 1, \qquad 0 \le \theta < 1, \tag{5.51}$$

[see Example (3), Section 4] then the same basic procedure leads to the following results: Let us write

$$\theta_j = \left[\theta + j\gamma\right] \bmod. 1 \qquad j = 0,1,2,\dots,n-1 \tag{5.52}$$

$$\theta_o = \theta.$$

Then we find

$$\langle (x)T_n^2U^n\rangle = \frac{(2ik)^2}{n^2}\Bigg\{ n\int_0^1 G^n(x,\theta)G^n(\theta,\theta)U^n(\theta)d\theta + 2(n-1)\int_0^1 G^n(x,\theta)G^n(\theta,\theta_1)U^n(\theta_1)d\theta$$

$$+ 2(n-2)\int_0^1 G^n(x,\theta)G^n(\theta,\theta_2)U^n(\theta_2)d\theta$$

$$+ \\ \vdots \tag{5.53}$$

$$+ 2\int_0^1 G^n(x,\theta)G^n(\theta,\theta_{n-1})U^n(\theta_{n-1})d\theta - n^2\int_0^1 G^n(x,\theta)G^n(\theta,\xi)U^n(\theta)d\theta d\xi\Bigg\},$$

and

$$< |(x)T_nU^n|^2> = \frac{(2k)^2}{n^2}\left\{ n\int_0^1 |G^n(x,\theta)|^2|U^n(\theta)|^2 d\theta + 2(n-1)\int_0^1 G^n(x,\theta)\overline{G^n(\theta,\theta_1)}U^n(\theta)\overline{U^n(\theta_1)}d\theta \right.$$

(5.54)

$$+ 2(n-2)\int_0^1 G^n(x,\theta)\overline{G^n(\theta,\theta_2)}U^n(\theta)\overline{U^n(\theta_2)}d\theta$$

$$+$$

$$\vdots$$

$$+ 2\int_0^1 G^n(x,\theta)\overline{G^n(\theta,\theta_{n-1})}\, U^n(\theta)\, \overline{U^n(\theta_{n-1})}\, d\theta$$

$$\left. - n^2 \left| \int_0^1 |G^n(x,\theta)|^2|U^n(\theta)|^2 d\theta \right| \right\} .$$

PART III. PROOFS

6. Proof of Theorem (1) and other results

6.1 Preliminary remarks. Our chief object here will be to prove the estimates on which Theorem (1) of Part I is based and some related results. In subsection (6.2) we shall derive a priori estimates of $\| U^n \|$, $\| u^n \|$ and of the maximum modulus $\|\!| G^n |\!\|$ of the radiating Green's function G^n [see (2.21)]. As we mentioned in Section 3, there are two types of estimates of the quantity $\| T_n^\nu U^n(x) \| \; / \; \| U^n(x) \|$ (and also of $\| T_n^\nu u^n(x) \| \; / \; \| u^n(x) \|$), the so-called T_n^ν-type and the $\hat{T}_n^\nu$-type estimates. The T_n-type, used in the proof of Theorem (1), will be derived in subsection (6.3). The $\hat{T}_n^\nu$-type estimate, a special case of which was employed in subsection (5.3) of Part II, will be derived in subsection (6.4).

6.2 A priori estimates for U^n, G^n and u^n

6.21 Estimates for $\| U^n \|$ and $\|\!| G^n |\!\|$. We shall prove here that

$$\| U^n(x) \| = \underset{-\infty < x < \infty}{\text{Maximum}} \; |U^n(x)| \leq \exp(n|z|/Z), \tag{6.1}$$

and

$$\|\!| G^n(x,x') |\!\| = \underset{-\infty<x,x'< \infty}{\text{Maximum}} \; |G^n(x,x')| \leq \exp(n|z|/Z). \tag{6.2}$$

Let $\phi_1(x)$ and $\phi_2(x)$ be solutions of the continuum differential equation

$$\frac{d^2\phi}{dx^2} + \left[k^2 + (iknz/Z)\right] \phi = 0, \qquad -\infty < x < \infty \, , \tag{6.3}$$

[cf. (2.20)] such that

$$\phi_1(x) = a_1 \exp(-ikx), \qquad x \leq 0, \tag{6.4}$$

$$\phi_2(x) = a_2 \exp(ikx), \qquad x \geq L, \tag{6.5}$$

where a_1 and a_2 are arbitrary constants. Then ϕ_1 and ϕ_2 may be expressed as follows:

$$\phi_1(x) = a_1 \exp(-ikx) - \frac{inz}{Z} \int_0^x \sin\left[k(x-y)\right] w(y)\phi_1(y)dy, \tag{6.6}$$

$$(6.7) \qquad \phi_2(L-x') = a_2\exp\left[ik(L-x')\right] - \frac{inz}{Z}\int_0^{x'} \sin\left[k(x'-y')\right]w(L-y')\phi_2(L-y')dy',$$

where (6.7) is obtained from the representation,

$$\phi_2(x) = a_2\exp(ikx) - \frac{inz}{Z}\int_L^{x} \sin\left[k(x-y)\right]w(y)\phi_2(y)dy,$$

after making the following substitutions

$$(6.8) \qquad x = L-x', \qquad y = L-y'.$$

From (6.6) and (6.7) we then have, since $w(y) \geq 0$,

$$(6.9) \qquad |\phi_1(x)| \leq |a_1| + (n|z|/Z)\int_0^{x} |\phi_1(y)|w(y)dy$$

$$(6.10) \qquad |\phi_2(L-x')| \leq |a_2| + (n|z|/Z)\int_0^{x'} |\phi_1(L-y')|w(L-y')dy'.$$

Now let

$$(6.11) \qquad f(x) \leq C + \int_0^{x} f(t)q(t)dt,$$

where $f(x)$ is a non-negative continuous function, $q(t)$ is a non-negative integrable function and C is a positive constant, then it can be shown [cf. [11] pp. 97-98] that

$$(6.12) \qquad f(x) \leq C\exp\left[\int_0^{x} q(t)dt\right].$$

Applying this lemma to (6.9) and (6.10) we get,

$$(6.13) \qquad |\phi_1(x)| \leq |a_1|\exp\left[(n|z|/Z)\int_0^{x} w(y)dy\right],$$

and

$$(6.14) \qquad |\phi_2(x)| \leq |a_2| \exp\left[(n|z|/Z) \int_x^L w(y)dy\right],$$

where we have reverted to the use of x and y in (6.14) [see (6.8)].

Now when $x \geq L$ the solution of the continuum equation U^n may be expressed as follows:

$$(6.15) \qquad U^n = t_n \exp(ikx),$$

where t_n is the transmission coefficient of the medium evaluated at $x = L$. But from the general theory of one-dimensional scattering [see [8]] it follows that

$$(6.16) \qquad 0 < |t_n| \leq 1.$$

Identifying U^n with ϕ_2 and t_n with a_2 in (6.13) we find, using the last relation in (6.16), that

$$|U^n(x)| \leq \exp\left[(n|z|/Z) \int_x^L w(y)dy\right] \leq \exp(n|z|/Z),$$

which leads immediately to (6.1).

The continuum Green's function $G^n(x,x')$ [see (2.21)] can be represented as follows:

$$(6.17) \qquad G^n(x,x') = \frac{1}{W(k)} \begin{cases} \phi_2(x',k)\ \phi_1(x,k), & x < x' \\ \phi_1(x',k)\ \phi_2(x,k), & x > x'. \end{cases}$$

Here, ϕ_1 and ϕ_2 satisfy the homogeneous equation (6.3) and $W(k)$ is the Wronskian of $\phi_1(x,k)$ and $\phi_2(x,k)$ - i.e.,

$$(6.18) \qquad W(k) = \phi_1(x,k) \frac{d\phi_2(x,k)}{dx} - \phi_2(x,k) \frac{d\phi_1(x,k)}{dx} .$$

Since $G^n(x,x')$ is radiating, ϕ_1 and ϕ_2 must behave as follows in the regions $x \leq 0$ and $x \geq L$:

$$(6.19) \qquad \phi_1(x,k) = t_n' \exp(-ikx), \qquad x \leq 0,$$

(6.20) $$\phi_2(x,k) = t_n \exp(ikx), \qquad x \geq L,$$

where, without loss of generality, we may take t_n and t_n' to be transmission coefficients* of the medium evaluated at $x = L$ and $x = 0$, respectively. In the region $x < 0$, ϕ_2 has the form,

(6.21) $$\phi_2(x,k) = \exp(ikx) + r_n \exp(-ikx),$$

r_n being the reflection coefficient. Substituting the expressions for ϕ_2 and ϕ_1 in (6.21) and (6.19), respectively, into (6.18) and evaluating the result at $x = 0$ we have (because $W(k)$ is independent of x):

(6.22) $$W(k) = 2ikt_n'.$$

Employing* (6.16), we find that

(6.23) $$|W(k)| \geq 2k > 0.$$

Combining this relation and (6.17) with (6.13) and (6.14) we obtain the sought-after result, namely (6.2).

6.22 Estimates for $||u^n||$. Here we shall prove that

(6.24) $$||u^n|| = \underset{-\infty < x < \infty}{\text{Maximum}} |u^n(x)| \leq \exp(n|z|/Z).$$

To this end, let the n scatterer positions be reordered in such a manner that

(6.25) $$x_1 > x_2 > \dots > x_n.$$

Let $u_j^n(x)$ denote the solution of (2.1) in the interval (x_{j+1}, x_j). Then within these intervals, $u_j^n(x)$ has the form

$$u_j^n(x) = a_j \exp(ikx) + b_j \exp(-ikx), \qquad x_{j+1} \leq x \leq x_j, \quad j = 1,\dots,n-1,$$

while in $x \geq x_1$ and $x \leq x_n$, $u^n(x)$ may be expressed as follows:

(6.26)
$$u_o^n(x) = a_o \exp(ikx) \qquad x \geq x_1,$$
$$u_n^n(x) = \exp(ikx) + b_n \exp(-ikx), \qquad x \leq x_n.$$

* It is known from the general theory that $|t_n| = |t_n'|$.

At the j^{th} scatterer we find, after invoking the jump conditions implicit in (2.1)*, and solving** for a_j and b_j in terms of a_{j-1} and b_{j-1}, that

$$(6.27)\qquad \begin{aligned} a_j &= a_{j-1} - (z/2Z)\left[a_{j-1} + \exp(-2ikx_j)b_{j-1}\right] \\ b_j &= b_{j-1} + (z/2Z)\left[a_{j-1} + \exp(2ikx_j)b_{j-1}\right] . \end{aligned}$$

Clearly then, for any j, we have

$$(6.28)\qquad |a_j| + |b_j| \le \left[1 + (|z|/Z)\right]\left[|a_{j-1}| + |b_{j-1}|\right] .$$

Consequently, we may conclude that

$$(6.29)\qquad |a_j| + |b_j| \le \left[1 + (|z|/Z)\right]^n |a_o|, \qquad j = 0,1,\ldots,n.$$

Now it is easy to verify that

$$(6.30)\qquad |a_j|^2 - |b_j|^2 = |a_{j-1}|^2 - |b_{j-1}|^2.$$

This relation implies

$$(6.31)\qquad |a_o|^2 = 1 - |b_n|^2 \le 1$$

and hence, in virtue of (6.29), that

$$(6.32)\qquad |a_j| + |b_j| \le \left[1 + (|z|/Z)\right]^n, \qquad j = 0,1,\ldots,n.$$

But, using (6.26) we find

$$|u_j^n| \le |a_j| + |b_j|, \qquad j = 0,1,\ldots,n ,$$

and consequently that

---

* Namely, $u^n(x_j+0) = u^n(x_j-0)$; $u_x^n\big|_{x=x_j+0} - u_x^n\big|_{x_j-0} = -\frac{ikz}{Z}u^n(x_j)$ [cf. footnote on first page of Part I].

** This is always possible since the determinant of the matrix of coefficients of a_j and b_j is $(-2ik) \neq 0$.

(6.33) $$||u^n|| \leq \underset{j=0,1,2,\ldots,n}{\text{Maximum}} \left\{|a_j| + |b_j|\right\} .$$

An immediate consequence of this relation is:

(6.34) $$||u^n|| \leq (1 + |z|/Z)^n.$$

But we know that

$$\log||u^n|| \leq n \log(1 + |z|/Z) \leq n|z|/Z,$$

the last relation being a result of the fact that $|z|/Z < 1$ |cf. (2.4)|. It follows immediately that

(6.35) $$||u^n|| \leq \exp(nz/Z),$$

which is the relation we set out to prove.

6.3 T_n^ν - <u>type estimates</u>. Our basic objectives here will be to prove the following two lemmas:

<u>Lemma (1)</u>. Let $f(x)$ be any function, continuous in the unit interval and let

(6.36) $$T_n^\nu f(x) = \int_0^L (2ik)G^n(x,x')T_n^{\nu-1}f(x)d\left[W^n(x)-W(x)\right], \qquad \nu = 1,2,\ldots,$$

[cf. (2.22)] where

(6.37) $$T_n^0 f(x) = f(x).$$

In addition, let γ_n, δ_n and ζ_n be defined as follows:

(6.38) $$\gamma_n = 1 + n|z| \, \| 2kG^n \| \, / \, 2Z,$$

(6.39) $$\delta_n = \text{Max.} \left\{1, \; n|z|/2Z\right\} ,$$

(6.40) $$\zeta_n = \left[16kh + \underset{0 \leq \xi \leq 1}{\text{l.u.b.}} \sum_{j=1}^{m-1} \xi \int_{\xi_j}^{\xi_{j+1}} d\left[W^n(x)-W(x)\right]\right] .$$

In (6.40), the points ξ_o, $\xi_1,\ldots,\xi_m$ where

(6.41) $$0 = \xi_o < \xi_1 < \xi_2 < \ldots < \xi_m = L,$$

constitute a subdivision of the interval into m subintervals of equal length

(6.42) $$h = \xi_{j+1} - \xi_j = L/m, \qquad kh < 1.$$

The point ξ, $0 \leq \xi \leq L$ is an additional variable point of subdivision introduced for technical reasons. As a superscript in the symbol $\sum^{\xi}$ ξ is used **as a reminder that it is an additional point of subdivision and that the sum depends on ξ.**

Conclusions: The norms $||T_n^{\nu}U^n||$ and $||T_n^{\nu}u^n||$ satisfy the following relations for $\nu = 1,2,\ldots$:

(6.43) $$||T_n^{\nu}U^n|| \leq \left(1 + \frac{n|z|}{2Z}\right) (\gamma_n\delta_n\zeta_n)^{\nu} \text{ Max.}\left\{1, ||U^n||\right\},$$

(6.44) $$||T_n^{\nu}u^n|| \leq \left(1 + \frac{n|z|}{2Z}\right) (\gamma_n\delta_n\zeta_n)^{\nu} \text{ Max.}\left[1, ||u^n||\right].$$

Remarks

R.1*: In (6.36) the upper and lower limits of the integral are to be interpreted as 1+0 and 0-0 respectively. Similarly, in (6.40) a ξ_j (ξ) appearing as a limit of integration is to be interpreted as ξ_j+0 ($\xi+0$) or as ξ_j-0 ($\xi-0$) according as it appears as an upper or lower limit. With this understanding, no ambiguity will arise in the interpretation of the above integrals when 0, 1, ξ_j, $j = 0,\ldots,m$, or ξ coincides with a scatterer position.

R.2: We have assumed in (6.42), and we shall assume hereafter, that $hk < 1$. This assumption is made merely for the sake of simplicity; the analysis which follows could be carried out with only minor modifications if it is ignored.

R.3: Note that the number m (≥ 1) of subdivisions is arbitrary; use will be made of this fact in the proof of Lemma (2) which we now state.

Lemma (2). Let $\zeta_n = \zeta_n(m)$ be defined as in (6.40) and let

(6.45) $$1 \geq \epsilon(n) > |W^n(x) - W(x)|.$$

---

* Read as Remark 1.

Conclusion:

$$\underset{1 \leq m < \infty}{\text{Minimum}} \left[\zeta_n(m) \right] \leq 2 \text{ Min.} \left\{ 1, 6\sqrt{2\epsilon(n) \text{Max.}\{kL,1\}} \right\} . \tag{6.46}$$

R.4: If we define β_n to be

$$\beta_n = \text{Min.} \left\{ 1, 6\sqrt{2\epsilon(n) \text{Max.}\{kL,1\}} \right\} \tag{6.47}$$

then (6.46) reduces to

$$\underset{1 \leq m < \infty}{\text{Minimum}} |\zeta_n(m)| \leq 2\beta_n, \tag{6.48}$$

and (6.43) and (6.44) become, since these relations hold for all $m \geq 1$,

$$\|T_n^\nu U^n\| \leq \left(1 + \frac{n|z|}{2Z}\right) (2\gamma_n \delta_n \beta_n)^\nu \text{ Max.} \left[1, \|U^n\|\right], \qquad \nu = 1,2,\ldots \tag{6.49}$$

$$\|T_n^\nu u^n\| \leq \left(1 + \frac{n|z|}{2Z}\right) (2\gamma_n \delta_n \beta_n)^\nu \text{ Max.} \left[1, \|u^n\|\right], \qquad \nu = 1,2,\ldots . \tag{6.50}$$

R.5: β_n as defined in (6.47) is identical with the β_n employed in Theorem (1) (cf. (3.3)).

In the following we shall assume the validity of Lemma (1) and give a proof of Lemma (2). Then we shall prove Lemma (1).

6.31 Proof of Lemma (2). From (6.40) it follows immediately that

$$\zeta_n = \zeta_n(m) \leq \frac{16kL}{m} + 2(m+1) \underset{0 \leq \xi \leq 1}{\text{l.u.b.}} \left| W^n(\xi) - W(\xi) \right| \tag{6.51}$$

$$\leq \frac{16kL}{m} + 2(m+1)\, \epsilon(n) \equiv B(m).$$

Let m_o be that value at which $B(m)$ achieves its minimum and let m_o' be the least integer equal to or exceeding m_o. Then, since the relation (6.51) holds for all integer values of m we find that

$$(6.52)\qquad \zeta_n(m_o') \leq B(m_o') = \frac{16kL}{m_o'} + 2(m_o' + 1)\,\epsilon(n) \leq \frac{16kL}{m_o} + 2(m_o + 2)\,\epsilon(n).$$

But m_o is easily shown to be

$$(6.53)\qquad m_o = \sqrt{8kL/\epsilon(n)}\ ;$$

hence

$$(6.54)\qquad \zeta_n(m_o') \leq \sqrt{32kL\epsilon(n)} + 2\sqrt{8kL\epsilon(n)} + 4\epsilon(n)$$

$$\leq 8\sqrt{2kL\epsilon(n)} + 4\sqrt{\text{Max.}\{kL,1\}}\ \epsilon(n),$$

$$\leq 12\sqrt{\text{Max.}\{kL,1\}\,\epsilon(n)}\ .$$

To complete the proof of (6.45), it suffices to show that

$$(6.55)\qquad \zeta_n(m_o') \leq 2.$$

To this end we note that

$$(6.56)\qquad \zeta_n(m_o') \leq 16kh + 2,$$

because the total variation of the last term in the right-hand side of (6.40) is at most 2. But $h = L/m$ can be made arbitrarily small - thus (6.55) is a consequence of (6.56) and (6.46) follows from (6.55) and (6.54).

6.32 Proof of Lemma (1). The following norms will be employed in the ensuing analysis. Let $f(x)$ be a function continuous in the interval $0 \leq x \leq L$ except perhaps at finite number - $(j-1)$ say - of points where finite jumps are allowed. Let I_ν, $\nu = 1,2,\ldots,j$, be the open sub-intervals bounded by these points. Then by $\|f\|$ and $\|f\|_1$ we shall mean:

$$(6.57)\qquad \|f\| = \underset{\nu=1,2,\ldots,j}{\text{Maximum}}\ \underset{x\in I_\nu}{\text{l.u.b}} \Big[\,|f(x)|\,\Big],$$

$$(6.58)\qquad \|f\|_1 = \text{Max.}\Big\{1, \|f\|\Big\}\ .$$

Similarly, let $g(x,y)$ be a function defined in the square $0 \leq x,\ y \leq L$, and continuous everywhere except perhaps at a finite number of simple curves which divide the square into a finite number of disjoint domains D_ν, $\nu = 1, 2,\ldots,j$. Assume that $g(x,y)$ suffers at most a jump discontinuity at each boundary curve. Then by $\|g\|$ we shall mean:

$$(6.59) \qquad |||g||| = \underset{\nu=1,2,\ldots,j}{\text{Maximum}} \ \underset{x,y \in D_\nu}{\text{l.u.b}} \left\{ |g(x,y)| \right\} .$$

R.6: Although the notation is the same, these new norms differ from the norms employed earlier [cf. (2.28) and (6.2)]. Let us for the moment append an asterisk to these new norms to distinguish them from the old. Then, comparing (6.57) and (2.28) we find

$$(6.60) \qquad ||U^n||^* \leq ||U^n|| , \qquad ||u^n||^* \leq ||u^n|| .$$

For $G^n(x,x')$ and $T_n^\nu U^n(x)$ and $T_n^\nu u^n(x)$, $\nu = 1,2,\ldots$, the corresponding relations are [cf. (6.59) and (6.2)]:

$$(6.61) \qquad |||G^n(x,x')|||^* = |||G^n(x,x')|||$$

$$(6.62) \qquad ||T_n^\nu U^n||^* = ||T_n^\nu U^n|| , \qquad ||T_n^\nu u^n||^* = ||T_n^\nu u^n||, \qquad \nu = 1,2,\ldots .$$

Here we have equality because $T_n^\nu(x)$ and $G^n(x,x')$ are proportional to $\exp(\pm ikx)$ and $\exp(\pm ikx \pm ikx')$ outside the unit interval and unit square, respectively.

R.7: We need only prove Lemma (1) in terms of the new norms; for if Lemma (1) holds for the norms $||\cdot||^*$ and $|||\cdot|||^*$, it follows immediately from (6.60) - (6.62) that it holds for $||\cdot||$ and $|||\cdot|||$.

R.8: Hereafter, we shall work exclusively with $||\cdot||$ and $|||\cdot|||$ as defined in (6.57) and (6.59).

The first step in our proof of Lemma (1) will be to show that

$$(6.63) \qquad |T_n^\nu f(x)| \leq \gamma_n |S_n^\nu f(x)|$$

where

$$(6.64) \qquad S_n^\nu f(x) = \int_0^L \exp\left[ik|x-x'|\right] T_n^{\nu-1} f(x') \, dF^n(x') ;$$

and

$$(6.65) \qquad F^n(x') = W^n(x') - W(x') .$$

To this end, we observe that $G^n(x,x')$, the radiating Green's function of the continuum problem, may be represented as follows:

$$G^n(x,x') = -\frac{\exp\left[ik|x-x'|\right]}{2ik} + \frac{nz}{2Z}\int_0^L \exp\left[ik(x-\xi)\right]G^n(\xi,x')dW(\xi). \tag{6.66}$$

Since $G^n(x,x')$ is a symmetric function of x and x' [see (6.17)], x and x' may be interchanged in the right-hand side of the above equation. The result of this interchange is:

$$G^n(x,x') = -\frac{\exp\left[ik|x-x'|\right]}{2ik} + \frac{nz}{2Z}\int_0^L \exp\left[ik(x'-\xi)\right]G^n(\xi,x)dW(\xi). \tag{6.67}$$

Substituting this expression into (6.36), the defining equation of $T_n^\nu f(x)$, we find

$$T_n^\nu f(x) = S_n^\nu f(x) + \frac{nz}{2Z}\int_0^L\left\{\int_0^L \exp\left[ik|x'-\xi|\right]2ikG^n(\xi,x)dW(\xi)\right\} T_n^{\nu-1}(x')fdF^n(x'). \tag{6.68}$$

After interchanging orders of integration we get

$$T_n^\nu f(x) = S_n^\nu f(x) + \frac{nz}{2Z}\int_0^L 2ik\, G^n(\xi,x)S_n^\nu f(\xi)dW(\xi). \tag{6.69}$$

Since the total variation of $W(\xi)$ is unity we have, from (6.69),

$$|T_n^\nu f(x)| \leq \left(1 + \frac{n|z|}{2Z}\,\|2kG^n\|\right)\;|S_n^\nu f(x)| = \gamma_n|S_n^\nu f(x)|,$$

which is what we set out to prove.

The problem of estimating $|T_n^\nu f(x)|$ is thus reduced to that of estimating $|S_n^\nu f(x)|$. To estimate $S_n^\nu f(x)$ we proceed as follows: Let

$$M_n^{\nu-1}(x,x') = \exp\left[ik|x-x'|\right]\; T_n^{\nu-1}f(x'). \tag{6.70}$$

Then we have, employing (6.65),

$$(6.71)\qquad |S_n^{\nu}f(x)| = \left| \int_0^L M^{\nu-1}(x,x')dF^n(x')\right|$$

$$\leq \left| \int_0^{\xi} M_n^{\nu-1}(x,x')dF^n(x')\right| + \left| \int_{\xi}^{L} M^{\nu-1}(x,x')dF^n(x')\right|$$

$$\leq \sum_{j=o}^{m-1}{}^{\xi} \left| \int_{\xi_j}^{\xi_{j+1}} \left[M_n^{\nu-1}(x,x')-M_n^{\nu-1}(x,\xi_j)\right]dW^n(x')\right|$$

$$+ \sum_{j=o}^{m-1}{}^{\xi} \left| \int_{\xi_j}^{\xi_{j+1}} \left[M_n^{\nu-1}(x,x')-M_n^{\nu-1}(x,\xi_j)\right]dW(x)\right|$$

$$+ \sum_{j=o}^{m-1}{}^{\xi} \left|M_n^{\nu-1}(x,\xi_j)\right| \left| \int_{\xi_j}^{\xi_{j+1}} dF^n \right| .$$

Here, ξ, $0 \leq \xi \leq L$, is the variable point of the subdivision mentioned above.

From the fact that the total variation of both $W^n(x)$ and $W(x)$ is unity we conclude that,

$$(6.72)\qquad |S_n^{\nu}f(x)| \leq 2\left\Vert M_n^{\nu-1}(x,x')-\widetilde{M}^{\nu-1}(x,x')\right\Vert + \left\Vert \widetilde{M}^{\nu-1}(x,x')\right\Vert \sum_{j=o}^{m-1}{}^{\xi} \left| \int_{\xi_j}^{\xi_{j+1}} dF^n(x')\right| ,$$

where $\widetilde{M}_n^{\nu-1}(x,x')$ is the function of two variables in the square, $0 \leq x,x' \leq L$ which varies like $M_n^{\nu-1}(x,\xi_j)$ in the strip $0 \leq x \leq L$, $\xi_j \leq x' \leq \xi_{j+1}$, $j = 0,1,2,\ldots,m-1$. Employing the defining equation of $M_n^{\nu-1}(x,x')$ [see (6.70)] we find that

$$(6.73)\qquad \left\Vert \widetilde{M}_n^{\nu-1}(x,x')\right\Vert \leq \left\Vert M_n^{\nu-1}(x,x')\right\Vert \leq \| T_n^{\nu-1}f(x)\|$$

$$(6.74)\qquad \left\Vert M_n^{\nu-1}(x,x')-\widetilde{M}_n^{\nu-1}(x,x')\right\Vert \leq \left\Vert \exp\left[ik|x-x'|\right]-\widetilde{\exp}\left[ik|x-x'|\right]\right\Vert \; \| T_n^{\nu-1}f(x)\|$$

$$+ \| T_n^{\nu-1}f(x)-\widetilde{T}_n^{\nu-1}f(x)\|^{\xi} .$$

Here, $\widetilde{\exp}\left[ik|x-x'|\right]$ is related to $\exp\left[ik|x-x'|\right]$ in the same way that $M_n^{\nu-1}(x,x')$ is related to $\widetilde{M}_n^{\nu-1}(x,x')$ and $\widetilde{T}_n^{\nu-1}f(x)$ is defined as the piecewise constant function of x in the interval $0 \leq x \leq L$ which assumes the constant values $T_n^{\nu-1}f(\xi_j)$ or $T_n^{\nu-1}f(\xi)$ in the sub-intervals having ξ_j or ξ, respectively as lower endpoints. In the last term of (6.74) the superscript ξ is employed to indicate the dependence of this quantity $\|\cdot\|$ on ξ.

Now it is easily verified that

$$(6.75) \qquad \left\| \exp\left[ik|x-x'|\right] - \widetilde{\exp}\left[ik|x-x'|\right] \right\| \leq kh, \qquad kh < 1,$$

where h is defined in (6.42). Consequently, employing (6.72)-(6.75) we find that

$$(6.76) \qquad |S_n^{\nu}f(x)| \leq \left[2kh + \sum_{j=o}^{m-1}{}^{\xi} \left| \int_{\xi_j}^{\xi_{j+1}} dF^n(x') \right| \right] \|T_n^{\nu-1}f\| + 2\| T_n^{\nu-1}f(x) - \hat{T}_n^{\nu-1}f(x)\|^{\xi}$$

which in virtue of (6.63) becomes

$$(6.76') \qquad |S_n^{\nu}f(x)| \leq \gamma_n \left[2kh + \sum_{j=o}^{m-1}{}^{\xi} \left| \int_{\xi_j}^{\xi_{j+1}} dF^n(x') \right| \right] \|S_n^{\nu-1}f\| + 2\| T_n^{\nu-1}f(x) - \hat{T}^{\nu-1}f(x)\|^{\xi}.$$

Now define $N\left[S_n^{\nu}f\right]$ as follows:

$$(6.77) \qquad N\left[S_n^{\nu}f\right] = \gamma_n \left[2kh + \underset{0 \leq \xi \leq 1}{\text{l.u.b}} \left\{ \sum_{j=o}^{m-1}{}^{\xi} \left| \int_{\xi_j}^{\xi_{j+1}} dF^n(x') \right| \right] \| S_n^{\nu-1}f\| \right.$$
$$+ 2 \underset{0 \leq \xi \leq 1}{\text{l.u.b}} \left\{ \| T_n^{\nu-1}f(x) - \hat{T}_n^{\nu-1}f(x)\|^{\xi} \right\} .$$

Clearly,

$$(6.78) \qquad |S_n^{\nu}f(x)| \leq \|S_n^{\nu}f\| \leq N\left[S_n^{\nu}f\right].$$

Our aim now is to estimate the last term in (6.77). We shall prove that*

$$(6.79) \qquad \underset{0 \leq \xi \leq 1}{\text{l.u.b}} \left\{ \| T_n^{\nu}f(x) - \hat{T}_n^{\nu}f(x)\|^{\xi} \right\} \leq 7kh\delta_n \gamma_n N\left[S_n^{\nu}f\right].$$

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* We assume here that $\nu \geq 2$. The case $\nu = 1$ will be treated later.

To this end we consider the quantity

$$(6.80)\qquad H_n^{\nu}(x,\eta) = T_n^{\nu}f(x+\eta) - T_n^{\nu}f(x), \qquad 0 \leq \eta \leq h,$$

where x is to be regarded as a fixed quantity ultimately to be identified with ξ_j, $j = 0,1,2,\ldots,m-1$ or ξ, and as indicated, η is a positive quantity not larger than the maximum length h of the intervals of subdivision. Using (6.80), (6.69) and (6.66) it is easy to verify that

$$(6.81)\qquad |H_n^{\nu}(x,\eta)| \leq |I_n^{\nu}(x,\eta)| + \delta_n\gamma_n|S_n^{\nu}f(x)|kh,$$

where

$$(6.82)\qquad I_n^{\nu}(x,\eta) \equiv S_n^{\nu}f(x+\eta) - S_n^{\nu}f(x),$$

and γ_n and δ_n are defined as in (6.38) and (6.39), respectively. Since the second term in (6.81) is already an estimate of the desired type [cf. (6.79)], we may limit our attention to obtaining one of like kind for I_n^{ν}. To this end, we employ the defining equation for $S_n^{\nu}f(x)$, (6.64), and rewrite $I_n^{\nu}(x,\eta)$ as follows:

$$(6.83)\qquad I_n^{\nu}(x,\eta) = \int_0^L \left\{\exp\left[ik\Delta(x';x,\eta)\right]-1\right\} \exp\left[ik|x-x'|\right]T_n^{\nu-1}f(x')dF^n(x'),$$

where

$$(6.84)\qquad \Delta(x';x,\eta) = |x+\eta-x'| - |x-x'| = \begin{cases} \eta, & 0 \leq x' \leq x, \\ -2(x'-x)+\eta, & x \leq x' \leq x+\eta, \\ -\eta, & x+\eta \leq x' \leq L. \end{cases}$$

Since

$$\exp\left[ik\Delta(x';x,\eta)\right]-1 = i\sin\left[k\Delta(x';x,\eta)\right]-2\sin^2\left[k\Delta(x';x,\eta)/2\right],$$

$I_n^{\nu}(x,\eta)$ may be expressed as the sum

$$I_n^\nu(x,\eta) = I_{n,1}^\nu(x,\eta) + I_{n,2}^\nu(x,\eta)$$

$$= i \int_0^L \sin\left[k\Delta(x';x,\eta)\right] \exp\left[ik|x-x'|\right] T_n^{\nu-1} f(x') dF^n(x')$$

(6.85)

$$- 2 \int_0^L \sin^2\left[k\Delta(x';x,\eta)\right] \exp\left[ik|x-x'|\right] T_n^{\nu-1} f(x') dF^n(x'),$$

where $I_{n,1}^\nu(x,\eta)$ denotes the first term and $I_{n,2}^\nu(x,\eta)$ is the second.

Consider the expression for $I_{n,1}^\nu(x,\eta)$; it may be recast in the following form:

$$\text{(6.86)} \qquad I_{n,1}^\nu(x,\eta) = i \int_0^L \sin\left[k\Delta(x';x,\eta)\right] d_{x'} \left(\int_0^{x'} \exp\left[ik|x-x'|\right] T_n^{\nu-1} f(y) dF^n(y) \right) .$$

After integrating by parts [see [12]. p.7] we get:

$$\text{(6.87)} \qquad I_{n,1}^\nu(x,\eta) = iS_n^\nu f(x) \sin\left[k\Delta(L;x,\eta)\right] - i \int_0^L K_n^\nu(x,x') d_{x'} \left\{ \sin\left[k\Delta(x';x,\eta)\right]\right\},$$

where

$$\text{(6.88)} \qquad K_n^\nu(x,x') = \int_0^{x'} \exp\left[ik|x-y|\right] T_n^{\nu-1} f(y) \, dF^n(y) \; .$$

Note that

$$\text{(6.89)} \qquad |K_n^\nu(x,x')| \le \left| \int_0^{x'} \exp\left[ik|x-y|\right] T_n^{\nu-1} f(y) \, dF^n(y) \right|$$

$$+ \left| \int_{x'}^L \exp\left[ik|x-y|\right] T_n^{\nu-1} f(y) \, dF^n(y) \right| \le N\left[S_n^\nu f(x)\right],$$

as can be verified by identifying x' with ξ in the second relation of (6.71) and taking the defining equation of $N\left[S_n^\nu f(x)\right]$ into account [see (6.77)]. Employing (6.87) - (6.89) we conclude that

$$\text{(6.90)} \qquad |I_{n,1}^\nu(x,\eta)| \le N\left[S_n^\nu f(x)\right] \; |\sin\left[k\Delta(L;x,\eta)\right]| \;\; N\left[S_n^\nu f(x)\right] V \left\{ \sin\left[k\Delta(x';x,\eta)\right] \right\}$$

where $V\{f(x')\}$ is the total variation of $f(x')$ on the unit interval. Since $\sin(x)$ is monotonic for $-1 \leq x \leq 1$ and [see (6.84)]

$$\left|\sin\left[k\Delta(1;x,\eta)\right]\right| \leq k\eta < kh, \qquad kh < 1,$$

it follows that

$$V\left\{\sin\left[k\Delta(x';x,\eta)\right]\right\} \leq V\left\{k\Delta(x';x,\eta)\right\} \leq 2k\eta \leq 2kh, \qquad kh \leq 1.$$

Hence we find, employing (6.90),

$$(6.91) \qquad |I^{\nu}_{n,1}(x,\eta)| \leq 3kh\, N\left[S^{\nu}_{n}f\right].$$

Similarly, it can be shown that*

$$(6.92) \qquad |I^{\nu}_{n,2}(x,\eta)| \leq 3kh\, N\left[S^{\nu}_{n}f\right], \qquad kh \leq 1$$

and consequently, employing (6.81), (6.82), (6.78), (6.85) and (6.91) that

$$(6.93) \qquad |H^{\nu}_{n}(x,\eta)| \leq 7kh\, \delta_n\gamma_n\, N\left[S^{\nu}_{n}f\right].$$

If we now identify x with the ξ_j, $j = 0,1,2,\ldots,m-1$ or ξ we find, employing (6.80), that

$$(6.94) \qquad \|\, T^{\nu}_{n}f(x) - \hat{T}^{\nu-1}f(x)\,\|^{\xi} \leq 7kh\, \delta_n\gamma_n\, N\left[S^{\nu}_{n}f\right].$$

The desired result, namely (6.79), follows immediately from this relation.

Returning to (6.77), and making use of the inequalities (6.7) and (6.79) we obtain the relation

$$(6.95) \qquad N\left[S^{\nu}_{n}f\right] \leq \delta_n\gamma_n\left[16kh + \underset{0\leq\xi\leq 1}{\text{l.u.b}}\left\{\sum_{j=0}^{m-1}{}^{\xi}\left|\int_{\xi_j}^{\xi_{j+1}} dF^{n}(x')\right|\right\}\right]N\left[S^{\nu-1}_{n}f\right]$$

which leads immediately to

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* This estimate is of course not the sharpest estimate obtainable.

$$(6.96)\qquad N\left[S_n^{\nu}f\right] \leq (\delta_n\gamma_n)^{\nu-1}\left[16kh + \underset{0\leq\xi\leq 1}{\mathrm{l.u.b}}\left\{\sum_{j=o}^{m-1}{}^{\xi}\left|\int_{\xi_j}^{\xi_{j+1}} dF^n(x')\right|\right\}\right]^{\nu-1} N\left[S_n^1 f\right],$$

and hence, employing (6.40) and (6.63) to

$$(6.97)\qquad \|T_n^{\nu}f\| \leq (\delta_n\gamma_n)^{\nu}\,\zeta_n^{\nu-1}\,N\left[S_n^1 f\right] .$$

The proof of the basic relations (6.43) and (6.49) will be complete provided we can show that

$$(6.98)\qquad N\left[S_n^1 U^n\right] \leq \zeta_n\left(1+\frac{nz}{2Z}\right)\|U^n\|_1$$

$$(6.99)\qquad N\left[S_n^1 u^n\right] \leq \zeta_n\left(1+\frac{nz}{2Z}\right)\|u^n\|_1 .$$

The desired estimate for $N\left[S_n^1 U^n\right]$ may be obtained as follows. In accordance with (6.37) U^n may be identified with $T_n^o U^n$. Employing (6.76) we find after, replacing f by U^n,

$$(6.100)\qquad |S_n^1 U^n(x)| \leq \left[2kh + \sum_{j=o}^{m-1}{}^{\xi}\left|\int_{\xi_j}^{\xi_{j+1}} dF^n(x')\right|\right]\|U^n\| + 2\|U^n(x)-\hat{U}^n(x)\|^{\xi} .$$

Here, $\hat{U}^n(x)$ is the piecewise constant function in the unit interval which assumes the constant values $U^n(\xi_j)$ or $U^n(\xi)$ in the subintervals having ξ_j or ξ, respectively, as lower endpoints. Now, using (2.19) it is easy to verify that

$$(6.101)\qquad |U^n(x+\eta)-U^n(x)| \leq \left[\exp(ik\eta)-1\right] + \frac{n|z|}{2Z}\left|\int_0^1\left\{\exp\left[ik|x'+-x|\right]-\exp\left[ik|x-x'|\right]\right\}\cdot U^n(x')dW(x')\right|$$

$$\leq kh\left(1+\frac{n|z|}{2Z}\right)\|U^n\|_1 ,$$

where η is the same variable employed above [see neighborhood of (6.73)] and where $\|U^n\|_1$ is defined as in (5.58).

After identifying x with ξ_j, $j = 0,1,\ldots,m-1$, we find

$$(6.102)\qquad \|U^n(x+\eta) - U^n(x)\|^{\xi} < \left(1+\frac{nz}{2Z}\right)kh\,\|U^n\|_1$$

which together with (6.100) yields the desired relation, namely (6.99).

If U^n is replaced by u^n in the above proof and the integral equation (2.17) is employed in place of (2.19) we find, in a similar manner, that (6.99) is valid. The basic relations (6.43) and (6.44) follow immediately from (6.97), (6.98) and (6.99).

6.4 $\hat{T}_n^\nu$-type estimates. Let $T_n^\nu f$ be defined as in (6.36) and (6.37) and as before let

$$F^n(x) = W^n(x) - W(x).$$

We shall show that

$$\| T_n^o U^n \| = \| U_n \| ,$$

$$\| T_n U^n \| \le \chi_n \| U_n \|_1 ,$$

(6.103)
$$\| T_n^{2\nu+1} U^n \| \le 2 \chi_n^{\nu+1/2} \lambda_n^\nu \sqrt{1+\chi_n/4} \; \| U_n \|_1 , \qquad \nu = 1,2,\ldots ,$$

$$\| T_n^{2\nu} U^n \| \le 2 \chi_n^\nu \lambda_n^\nu \| U_n \|_1 , \qquad \nu = 1,2,\ldots ,$$

where

(6.104)
$$\chi_n = \hat{\delta}_n \hat{\gamma}_n \hat{\zeta}_n ,$$

$$\hat{\delta}_n = \text{Max.} \left\{ 1, 2k \| G^n \| \right\} ,$$

(6.105)
$$\hat{\gamma}_n = 1 + (nz\hat{\delta}^n/2Z) ,$$

$$\hat{\zeta}_n = 4kh + \sum_{j=o}^{m} \left| \int_{\xi_j}^{\xi_{j+1}} dF^n(x') \right| .$$

Here, the quantities $\xi_o, \ldots, \xi_{m+1}$,

(6.106)
$$0 = \xi_o < \xi_1 < \ldots < \xi_m \le \xi_{m+1} = L,$$

are m+2 points which subdivide the scattering interval into m subintervals of equal length

(6.107)
$$h = \xi_{j+1} - \xi_j ,$$

and λ_n is given by

$$(6.108) \qquad \lambda_n = 1 + \chi_n/2 + \sqrt{\chi_n + (\chi_n/2)^2} .$$

The corresponding estimates for $T_n^{\nu} u^n(x)$, $\nu = 0,1,2,\ldots$, may be obtained from (6.103) simply by replacing U^n by u^n.

R.9: This result is the analog of Lemma (1) of subsection (6.3). Comparing the expression of $\hat{\zeta}_n$ in (6.105) with ζ_n in (6.40) we note the absence in the former of the variable point of subdivision ξ and of the associated least-upper-bound operation. It is the absence of this operation that makes the $\hat{T}_n^{\nu}$-estimate better suited, in some respects, for dealing with statistical problems. (In such a setting, we may add, it is convenient to replace $\hat{\zeta}_n$ by 2 [see R.10 below] in $\sqrt{1 + \chi_n}$ and wherever $\hat{\zeta}_n$ appears in the expression for λ_n. When this is done only the factors χ_n^{ν} and $\chi_n^{\nu + 1/2}$ in (6.103) depend upon the stochastic parameter θ). For example, in Part II [see neighborhood of (5.33)] it was ultimately the absence of the l.u.b. which made it possible to evaluate $< (\epsilon(n;x,\theta))^2 >$ explicitly in terms of n.

R.10: The analog of Lemma (2) in subsection (6.3) is:

$$(6.109) \qquad \underset{1 \leq m < \infty}{\text{Minimum}} \left[\hat{\zeta}_n(m) \right] \leq 2\hat{\beta}_n \equiv 2\text{Min.}\left[1, \sqrt{\text{Max.}\{1,kL\}\epsilon(n;x)} \right] ,$$

where

$$(6.110) \qquad \epsilon(n;x) = W^n(x) - W(x).$$

The proof is exactly analogous to the one used to prove Lemma (2).

R.11: The quantity $\hat{\beta}_n$ defined above corresponds to β_n in the T_n^{ν}-type estimate [cf. (6.47)]. Both $\hat{\beta}_n$ and β_n become small when n is large. Note, however, that the $\hat{T}_n^{\nu}$-type estimates are $O\left[(\hat{\beta}_n)^{\nu/2}\right]$ for $\nu \geq 2$ while the T_n^{ν}-type estimates are $O\left[(\beta_n)^{\nu}\right]$ for $\nu \geq 2$. If n is sufficiently large, then, provided the remaining factors in the $\hat{T}_n^{\nu}$ and T_n^{ν}-type estimates are comparable (this will be the case when nr is sufficiently small), the T_n^{ν}-type estimates will be more accurate than the $\hat{T}_n^{\nu}$-type. It was with this fact in mind that we spoke earlier of the T_n^{ν}-type estimates being more accurate in some respects than the $\hat{T}_n^{\nu}$-type estimates.

We turn now to the proof of the relations (6.103). Let us write

$$P_n^{\alpha-1}(x,x') = 2ikG^n(x,x')T_n^{\alpha-1}U_n(x'). \tag{6.111}$$

Then $T_n^{\alpha}U^n(x)$ may be written as follows:

$$\begin{aligned} T_n^{\alpha}U^n(x) = & \sum_{j=o}^{m} \int_{\xi_j}^{\xi_{j+1}} \left[P_n^{\alpha-1}(x,x')-P_n^{\alpha-1}(x,\xi_j)\right] dW^n(x') \\ & - \sum_{j=o}^{m} \int_{\xi_j}^{\xi_{j+1}} \left[P_n^{\alpha-1}(x,x')-P_n^{\alpha-1}(x,\xi_j)\right] dW(x') \\ & + \sum_{j=o}^{m} P_n^{\alpha-1}(x,\xi_j) \int_{\xi_j}^{\xi_{j+1}} dF^n(x'). \end{aligned} \tag{6.112}$$

From (6.111), (6.112) and (6.36) it follows easily that*

$$\begin{aligned} \| T_n^{\alpha}U^n(x) \| \leq & \ 4k \| T_n^{\alpha-1}U^n \| \ \| G^n(x,x')-\tilde{G}^n(x,x') \| \\ & + 2k \| G^n(x,x') \| \ \| T_n^{\alpha-1}U^n \| \left\{ \sum_{j=o}^{m} \left| \int_{\xi_j}^{\xi_{j+1}} dF^n(x') \right| \right\} \\ & + 4k \| G^n(x,x') \| \ \| T_n^{\alpha-1}U^n(x) - \tilde{T}_n^{\alpha-1}U^n(x) \| , \end{aligned} \tag{6.113}$$

where $\tilde{G}^n(x,x')$ is defined as that function of x and x' in the square $0 \leq x,x' \leq L$ which has the values $G^n(x,\xi_j)$ in the strip $\xi_j \leq x' \leq \xi_{j+1}$, $0 \leq x \leq L$, $j = 0,1,\ldots,m$ and $\tilde{T}_n^{\alpha-1}U^n(x)$ is the piecewise constant function which assumes the values $T_n^{\alpha-1}U_n(\xi_j)$ in the intervals $\xi_j \leq x \leq \xi_{j+1}$, $j = 0,1,\ldots,n$.

Now it follows directly from (6.36) and (6.37) that

$$\| T_n^{o}U^n(x) - \tilde{T}_n^{o}U^n(x) \| = \| U^n(x) - \tilde{U}^n(x) \| \tag{6.114}$$

and

---

* As in subsection 6.3 we shall assume that $kh < 1$. See, in this connection, remark R.2 on p. 47 which are also relevant here.

$$(6.115)\qquad ||T_n^{\alpha-1}U^n(x) - \tilde{T}_n^{\alpha-1}U^n(x)|| \le 2k||T_n^{\alpha-2}U^n(x)||\,\|G^n(x,x')-\tilde{G}^n(x,x')\|\,,\quad \alpha \ge 2,$$

where $\tilde{U}^n(x)$ is the piecewise constant function which assumes the values $U^n(\xi_j)$ in the intervals $\xi_j \le x \le \xi_{j+1}$. Employing (2.19) and (6.66) we find that

$$(6.116)\qquad \begin{aligned} &||U^n(x)-\tilde{U}(x)|| \le kh\left[1 + (nz/2Z)\right]||U^n||_1 \\ &\|G^n(x,x')-\tilde{G}^n(x,x')\| \le (h/2)\left[1 + (nz/2Z)\,\|2kG^n(x,x')\|\right]. \end{aligned}$$

Combining (6.114) - (6.116) and employing the notation in (6.105) we find:

$$(6.117)\qquad ||TU^n(x)|| \le \delta_n\gamma_n\zeta_n||U^n||_1$$

$$(6.118)\qquad \begin{aligned} ||T^\alpha U^n(x)|| &\le \delta_n\gamma_n\zeta_n\left\{||T_n^{\alpha-1}U^n(x)|| + ||T_n^{\alpha-2}U^n(x)||\right\} \\ &= \chi_n\left\{||T_n^{\alpha-1}U^n(x)|| + ||T_n^{\alpha-2}U^n(x)||\right\}, \qquad \alpha \ge 2. \end{aligned}$$

The relation (6.118) may be rewritten as follows:

$$(6.119)\qquad \begin{bmatrix} ||T_n^{2\nu+1}U^n|| \\ \\ ||T_n^{2\nu}U^n|| \end{bmatrix} \le \chi_n \begin{pmatrix} 1+\chi_n & \chi_n \\ \\ 1 & 1 \end{pmatrix} \begin{vmatrix} ||T_n^{2\nu-1}U^n|| \\ \\ ||T^{2\nu-2}U^n|| \end{vmatrix}, \quad \nu \ge 1,$$

which implies that

$$(6.120)\qquad \begin{bmatrix} ||T^{2\nu+1}U^n|| \\ \\ ||T_n^{2\nu}U^n|| \end{bmatrix} \le \chi_n^\nu \begin{pmatrix} 1+\chi_n & \chi_n \\ \\ 1 & 1 \end{pmatrix}^\nu \begin{bmatrix} ||T_n^1U^n||_1 \\ \\ ||T_n^0U^n||_1 \end{bmatrix}, \quad \nu \ge 1.$$

Introducing the eigenvectors of the matrix in the right-hand side of (6.119) namely,

$$\underline{x}^{+} = \begin{bmatrix} \lambda^{+} - 1 \\ 1 \end{bmatrix}, \qquad \underline{x}^{-} = \begin{bmatrix} \lambda^{-} - 1 \\ 1 \end{bmatrix} \tag{6.121}$$

where

$$\begin{aligned} \lambda^{+} &= 1 + \chi_n/2 + \sqrt{\chi_n + (\chi_n/2)^2} \\ \lambda^{-} &= 1 + \chi_n/2 - \sqrt{\chi_n + (\chi_n/2)^2}\ , \end{aligned} \tag{6.122}$$

are the eigenvalues that belong to $\underline{x}^{+}$ and $\underline{x}^{-}$ respectively, we find that

$$\begin{bmatrix} \|T_n^{2\nu+1}U^n\| \\ \\ \|T_n^{2\nu}U^n\| \end{bmatrix} \leq \chi_n^{\nu}(\lambda_n^{+})^{\nu} \begin{bmatrix} \sqrt{\chi_n + (\chi_n/2)^2} \\ \\ 2 \end{bmatrix} \|U\|_1, \qquad \nu = 1,2,\ldots$$

The last two relations of (6.103) follow immediately from this result. Moreover, if the integral equation (2.17) is employed in lieu of the integral equation (2.19), in the above analysis, it is easily verified that (6.103) is valid when U^n is replaced by u^n. This completes our derivation of the relations (6.103).

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Appendix I

IA. Transmission line model of the scattering process

The basic equations for a uniform non-dissipative transmission line are:

$$\frac{dI}{dx} = (i\omega c)\, V, \tag{I.1}$$

$$\frac{dV}{dx} = (i\omega \ell)\, I. \tag{I.2}$$

Here, I and V represent the current and voltage, respectively, at the point x, ℓ the series inductance per unit length and c the shunt capacitance per unit length. The harmonic time dependence $\exp(-i\omega t)$ has been assumed. When n small (identical) lumped impedances z are inserted into the line at $x_1, \ldots, x_n$ (see Figure 1), it is necessary to modify (I.2) as follows,

$$\frac{dV}{dx} = i\omega\ell I - z\left(\sum_{q=1}^{n} \delta(x-x_q)\right) I, \tag{I.3}$$

in order to take into account the voltage drop across these elements. Differentiating (I.1) with respect to x and combining the result with (I.3) we find that

$$\frac{d^2 I}{dx^2} + \left[k^2 + \frac{ikz}{Z} \sum_{q=1}^{n} \delta(x-x_q)\right] I = 0, \tag{I.4}$$

where we have set

$$\begin{aligned} k &= \omega\sqrt{\ell c}\ , \\ Z &= \sqrt{\ell/c}\ . \end{aligned} \tag{I.5}$$

If z is given as a function of ω it can be re-expressed as a function of k by means of (I.5). Z, as is well known, is the characteristic impedance of the line.

I.B. Mechanical model of the scattering process

The equation of motion of an infinite string undergoing small vibrations transverse to the x-axis is

$$(I.6) \qquad T\frac{\partial^2 U}{\partial x^2} = \rho\frac{\partial^2 U}{\partial t^2}, \qquad -\infty < x < \infty,$$

where $U = U(x,t)$ is the transverse displacement, and T and ρ are, respectively, the constant tension and mass density of the string. The characteristic impedance of the string is given by the following expression (cf. [13] pp. 125-9):

$$(I.7) \qquad Z = (T\rho)^{1/2}.$$

When scattering elements [see Figure 2] are present at $x = x_q$, $q = 1,2, \ldots,n$, (I.6) applies only for $x \neq x_q$. At $x = x_q$ we have, corresponding to (I.6), the following equation:

$$(I.8) \qquad T\left[\frac{dU}{dx}\right]^q = sU(x_q,t) + M\frac{d^2U(x_q,t)}{dt^2}, \qquad q = 1,2,\ldots,n.$$

Here, s is the spring constant of the scatterer, M the discrete mass. $[dU/dx]^q$ is defined as follows:

$$(I.9) \qquad \left[\frac{dU}{dx}\right]^q = \left.\frac{dU}{dx}\right|_{x_q-0} - \left.\frac{dU}{dx}\right|_{x_q+0}, \qquad q = 1,2,\ldots,n.$$

Assuming $U(x,t)$ to be of the form

$$(I.10) \qquad U(x,t) = u^n(x)\exp(-i\omega t),$$

and setting

$$(I.11) \qquad k = \omega(\rho/T)^{1/2},$$

$$(I.12) \qquad z = i\left[(s/\omega)-\omega M\right],$$

we find that (I.6) and (I.8) reduce respectively to

$$(I.13) \qquad \frac{d^2u^n}{dx^2} + k^2u^n = 0, \qquad x \neq x_q, \qquad q = 1,2,\ldots,n,$$

and

$$(I.14) \qquad \left[\frac{du^n}{dx}\right]^q = -(ikz)/Z, \qquad q = 1,2,\ldots,n.$$

The quantity z in (I.12) will be recognized as the impedance of a spring with an attached mass. Eliminating ω from (I.12) by means of (I.11), we obtain z as a function of k:

$$(I.15) \qquad z = i(\rho/T)^{1/2}\left[s/k - kMT/\rho\right].$$

Equations (I.13) and (I.14) may be combined to read:

$$(I.16) \qquad \frac{d^2u^n}{dx^2} + \left[k^2 + (ikz/Z)\sum_{q=1}^{n} \delta(x-x_q)u^n\right] = 0, \qquad -\infty < x < \infty.$$

This symbolic equation will be recognized as another instance of (2.1) of Part I.

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